NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS | WWW.NCTM.ORG

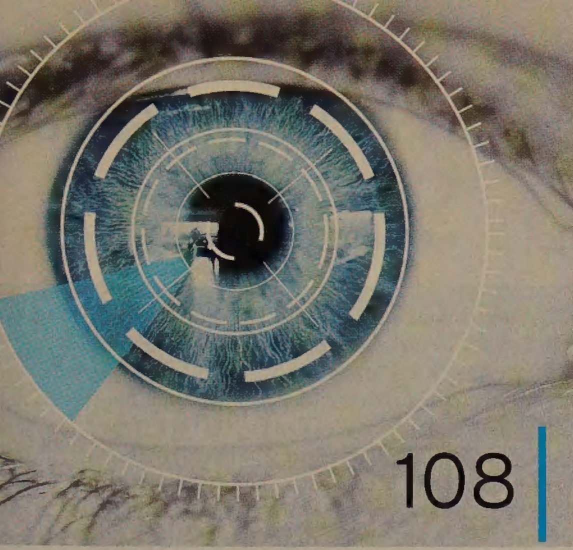
MATHEMATICS teacher

SEPTEMBER 2014

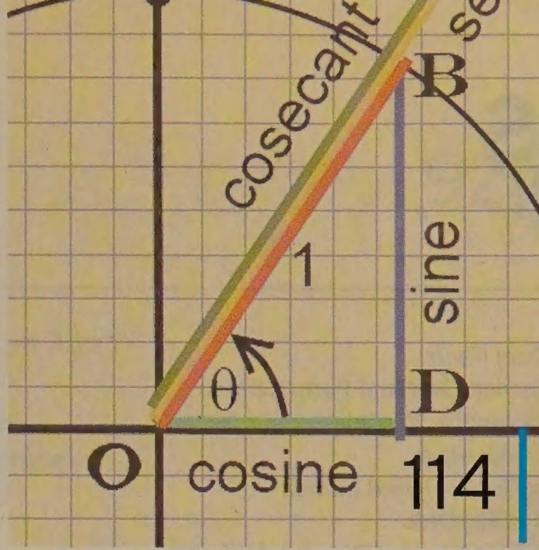
probability & perception

Posing and Solving **Converse Problems**

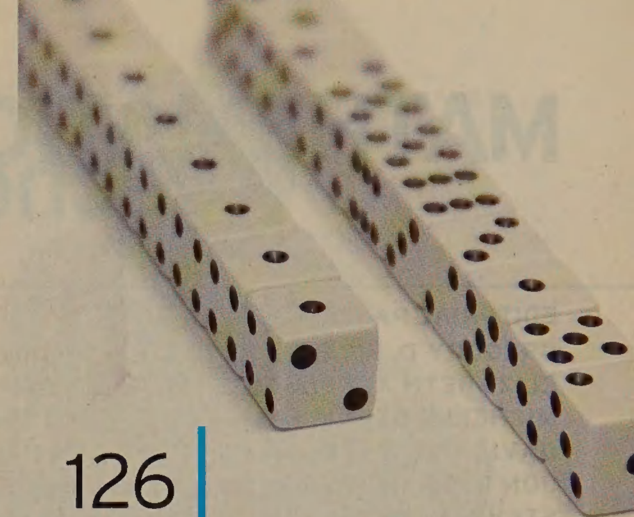
Focus, Coherence, and **Rigor** Frame the Standards



108



114



126

contents

Volume 108, Number 2, September 2014

features

on the cover

How do you decide which of two events is more probable? How do your students decide? In "Probability and Perception: The Representativeness Heuristic in Action" (pp. 126–31), authors Yun Lu, Francis J. Vasko, Trevor J. Drummond, and Lisa E. Vasko detail a situation in which experiment and experience help students relinquish faulty intuition about relative likelihood.

COVER: GARY SLUDDEN/GETTY IMAGES

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

To contact a journal staff person, e-mail **mt@nctm.org**.

Mission Statement: The National Council of Teachers of Mathematics is the public voice of mathematics education, supporting teachers to ensure equitable mathematics learning of the highest quality for all students through vision, leadership, professional development, and research.

Mathematics Teacher, an official NCTM journal, is devoted to improving mathematics instruction for grades 8–14 and supporting teacher education programs. It provides a forum for sharing activities and pedagogical strategies, deepening understanding of mathematical ideas, and linking mathematics education research to practice. *Mathematics Teacher* solicits submissions from high school mathematics teachers, university mathematicians, and mathematics educators and strongly encourages manuscripts in which ideas relate to classroom practice. Manuscripts previously published in other journals or books are not acceptable. NCTM publications present a variety of viewpoints. The views expressed or implied in this journal, unless otherwise noted, should not be interpreted as official NCTM positions. The appearance of advertising in NCTM's publications and on its websites in no way implies endorsement of approval by NCTM of any advertising claims or of the advertiser, its product, or its services. NCTM disclaims any liability whatsoever in connection with advertising appearing in NCTM's publications and on its websites.

All correspondence should be addressed to the *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Manuscripts should be prepared according to the *Chicago Manual of Style* and the United States Metric Association's *Guide to the Use of the Metric System*. No author identification should appear on the manuscript; the journal uses a blind-review process. To send submissions, access **mt.msubmit.net**. Send letters to the editor to **mt@nctm.org**.

For information on **article photocopies or back issues**, contact the Customer Care Department in the headquarters office.

A cumulative index appears on the NCTM website at **www.nctm.org/mt/mt-indexes.htm**. The *Mathematics Teacher* is indexed in *Biography Index, Contents Pages in Education, Current Index to Journals in Education, Education Index, Exceptional Child Education Resources, Literature Analysis of Microcomputer Publications, Mathematical Reviews, Media Review Digest,* and *Zentralblat für Didaktik der Mathermatik.*

Information is available from the Headquarters Office or online at www.nctm.org/membership regarding the three **other official journals**, *Teaching Children Mathematics, Mathematics Teaching in the Middle School*, and the *Journal for Research in Mathematics Education*. Dues support the development, coordination, and delivery of NCTM's services. Dues for individual membership are \$87 (U.S.), which includes \$37 for each *Mathematics Teacher* subscription. Each additional school journal (*Mathematics Teaching in the Middle School* and *Teaching Children Mathematics*) subscription is \$37. Each additional subscription to the *Journal for Research in Mathematics Education* is \$64. Foreign subscribers add \$18 (U.S.) postage for the first journal and \$4 (U.S.) postage for each additional journal. Special rates for students, institutions, bulk subscribers, and emeritus members are available from the Headquarters Office.

The *Mathematics Teacher* (ISSN 0025-5769) (IPM 1213431) is published monthly except June and July, with a combined December/January issue, by the National Council of Teachers of Mathematics at 1906 Association Drive, Reston, VA 20191-1502. Periodicals postage paid at Herndon, Virginia, and at additional mailing addresses. POSTMASTER: Send address changes to *Mathematics Teacher*, 1906 Association Drive, Reston, VA 20191-1502. Telephone: (703) 620-9840; orders: (800) 235-7566; fax: (703) 476-2970; e-mail: nctm@nctm.org.

departments

92

BLUFISH77/THINKSTOCK

calls for manuscripts

more4U

Look online for these additional items:

- "Unit Circles and Inverse Trigonometric Functions" (p. 114): Two worksheets for the unit circle
- "Probability and Perception: The Representativeness Heuristic in Action" (p. 126): An Excel file for a dice game
- "Interpolation and Polynomial Curve Fitting" (p. 132): Excel files for the Newton polynomial and the Lagrange polynomial

Download one of the free apps for your smartphone. Then scan this tag to access **www.nctm.org/mt.**

in the next issue

Coming in October 2014:

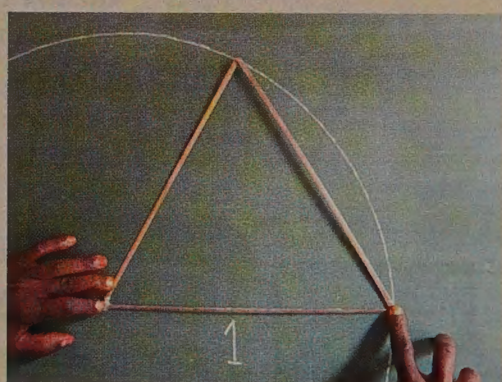
- "Lesson Planning with the Common Core," by Linda A. Estes, Amy Roth McDuffie, and Cathie Tate
- "Probability Explorations in a Multicultural Context," by Nirmala Naresh, Suzanne R. Harper, Jane M. Keiser, and Norm Krumpe
- "Martin Gardner: The Best Friend Mathematics Ever Had," by Colm Mulcahy and Albert Goetz

FLEXIBILITY IN TRIGONOMETRY

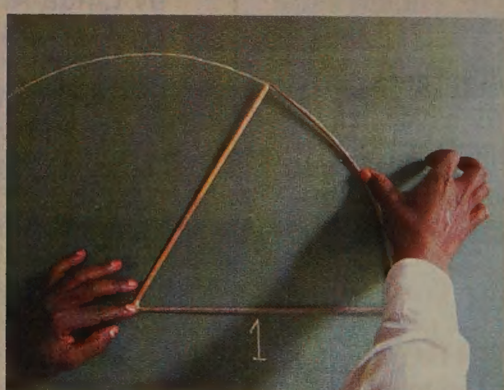
Two simple manipulatives can help students with trigonometry.

First, construct an equilateral triangle with some flexible material, such as bamboo strips (taped at the vertices) (see **fig. 1a [Metz]**). Stretch a side of the triangle so that it fits on the unit circle (see **fig. 1b [Metz]**). As the side is pulled, the 60° angle at the center of the circle closes, giving visual evidence that a radian is a bit less than 60°.

The second manipulative is a stick of flexible material (such as a thin strip of bamboo) of length $r\theta$. As a consequence of the fact that the area of a circular sector is given by $(1/2)\theta r^2$, with θ measured in radians, the right triangle *ABC* in **figure 2 (Metz)** has the same area as that of the sector. The sector can be drawn on the board. A stick of flexible material (such as a thin strip of bamboo) of length $r\theta$ is held at *B* and a string is attached to the stick at *C*. Students can pull or release this string from *A*. This demonstration helps students recall the formula for the area of a sector of a circle.



(a)



(b)

Fig. 1 (Metz)

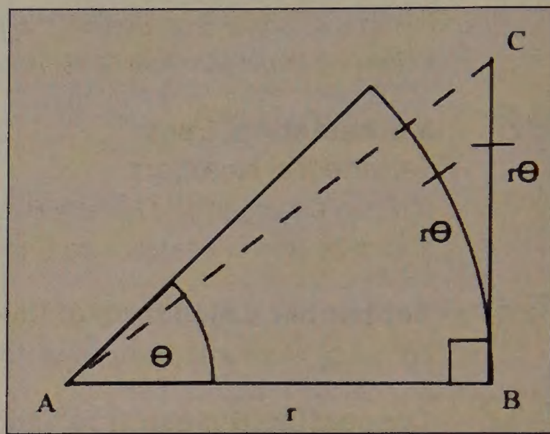


Fig. 2 (Metz)

James Metz
metz@hawaii.edu
Kapi'olani Community College
Honolulu, HI, Sept. 4, 2013

A CONTEMPORARY APPLICATION OF THE STABLE PAIRING ALGORITHM

Readers who enjoyed "The Stable Pairing Problem" (*MT* Feb. 2014, vol. 107, no. 6, pp. 446–50) might also be interested in a contemporary application as mentioned in Emma Brown's article in the *Washington Post*: "D.C. Preparing a New Unified Enrollment Lottery for Its Traditional and Charter Schools" (Nov. 19, 2013). The article, accessible at http://www.washingtonpost.com/local/education/dc-rolls-out-unified-enrollment-lottery-for-traditional-charter-schools/2013/11/19/448ee1e0-4ca7-11e3-9890-a1e0997fb0c0_story.html, explains that the District of Columbia will be implementing a technique very similar to that described by Greenwell so that students will gain acceptance at a school they wish to attend.

Albert Goetz
albert.goetz103@gmail.com
Adjunct, George Mason University
Fairfax, VA, Jan. 26, 2014

A VEHICLE FOR REASONING

Kudos to Michael Hardy ("Racing Ratios," *MT* April 2014, vol. 107, no. 8, pp. 587–91) for using scale models of cars as an authentic vehicle to inspire proportional reasoning. Because most of his activity can be completed with a scale model car that is not motorized, readers may also appreciate the activity presented in "Mathematical Explorations: Hot Wheels" (*Mathematics Teaching in the Middle School,* Nov. 2009, vol. 15, no. 4, pp. 239–43), which uses a more affordable and familiar vehicle.

As another example that students can critique when they discuss conversion factors, I would like to offer a lively problematic example (adapted from John Allen Paulos, *Innumeracy,* 1988). Consider the fraction (1/4 yd.)/(9 in.), which is "essentially a clever way of writing 1," as Hardy says in figure 3. Now, because the principal square root of 1 is still equal to 1, it might seem that (1/4 yd.)/(9 in.) should equal (1/2 yd.)/(3 in.), but this latter fraction is clearly *not* equal to 1. What happened?

Lawrence Lesser
lesser@utep.edu
The University of Texas at El Paso
Mar. 18, 2014

YOUR GREAT-GRANDFATHER'S ALGEBRA

Recently I joined a research group, the Algebra Textbook Project 1894–1930, which is investigating the treatment of applications in algebra in the United States at the turn of the last century. As a preliminary activity, we began looking at word problems in textbooks published around that time. I found this activity interesting and, in the end, useful. This task was greatly simplified by the fact that many textbooks from that period are available on Google Books®.

I was somewhat surprised to see how little the problems have changed over the years. Most of them could be printed almost verbatim in a modern textbook.

For example, although we rarely buy horses and carriages anymore, with a slight update the following problem could be used:

> A man bought a horse and carriage for $500, paying three times as much for the carriage as for the horse. How much did each cost? (Wallace Clarke Boyden, *A First Book in Algebra,* 1896, p. 8, no. 2)

At the time that the following problem was written, it was referring to a future event. If we change the tense from future to present, we could still use it:

> The Panama Canal will be 46 miles long. Of this distance the lower land parts on the Atlantic and Pacific sides will together be 9 times the length of the Culebra Cut, or hill part. How many miles long will the Culebra Cut be? Prove answer. (Joseph Victor Collin, *Practical Algebra, First Year Course,* 1910 [New York: American Book Company], p. 11, no. 14)

Back then, the authors looked to earlier works for problems too:

> Demochares has lived a fourth of his life as a boy; a fifth as a youth; a third as a man; and has spent 13 years in his dotage; how old is he? (From a collection of questions by Metrodorus, 310 CE) (Collins, 1910, p. 143, no. 6).

He is 60. Modern readers might be upset by the idea of entering their dotage at 47.

Finally, I found this gem. I have not seen this type of problem in a modern textbook, but perhaps I just haven't seen enough modern textbooks:

> The height of a certain flagstaff is unknown; but it is observed that a flag rope fastened to the top of the staff is two feet longer than the staff, and that its end just reaches the ground when carried to a point 18 feet distant from the foot of the staff. What is the height of the staff? (Henry Buchard Fine, *A College Algebra,* 1904 [Boston: Ginn and Company], p. 125, no. 15)

The problem is solved by applying the Pythagorean theorem to the right triangle formed by the flagstaff, the ground, and the extended rope. The resulting equation, $(x + 2)^2 = x^2 + 18^2$, simplifies to a linear equation since the square terms drop out. This equation is easily solved; the height of the flagstaff is 80 feet. This method is very practical. Most of the problems that I've seen involving heights of flagpoles depend on measuring shadows on sunny days or somehow finding angles of elevation. Here we need only two measurements that can be taken with a tape measure, rain or shine.

Looking through these century-old mathematics books can be a lot of fun. Challenging students to find and solve what they consider the most interesting problem can be a great contest or project.

Thomas J. Bannon
tbannon@adelphi.edu
Adelphi University
Garden City, NY, Feb. 11, 2014

PROBLEM 2, FEBRUARY 2014 CALENDAR

Problem 2 in the February 2014 Calendar (*MT* Feb. 2014, vol. 107, no. 6, p. 440) was stated as follows:

> Two points are chosen on the parabola defined by $y = x^2$, one with a positive x-coordinate and the other with a negative x-coordinate. If the points are (a, b) and (c, d), where $a < 0$ and $c > 0$, find the y-intercept of the line joining the two points in terms of a and c.

When I read the problem, my first instinct was to solve the problem in the way it was solved in the solutions section. Then I decided to take another look at the problem from the point of view of area. When factoring, the sum of cubes came about naturally in my solution, so I thought it might be worth sharing. (How often does that really happen?)

In the spirit of looking at problems in more than one way, here's another approach to this problem using area. Viewing this problem from a geometry standpoint, consider the trapezoid $WXYZ$ (see **fig. 1 [Sweeney]**) with distances marked. The problem now becomes a matter of expressing y in terms of a and c. The area of $WXYZ$ is given by the familiar formula for the area of a trapezoid:

$$A = \frac{1}{2}\left(a^2 + c^2\right)\left(-a + c\right)$$

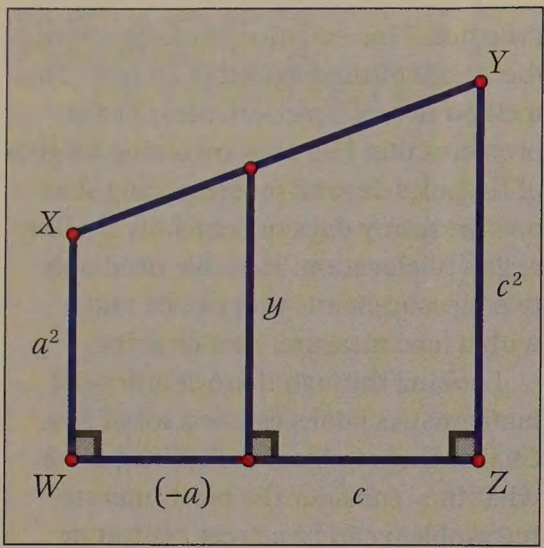


Fig. 1 (Sweeney)

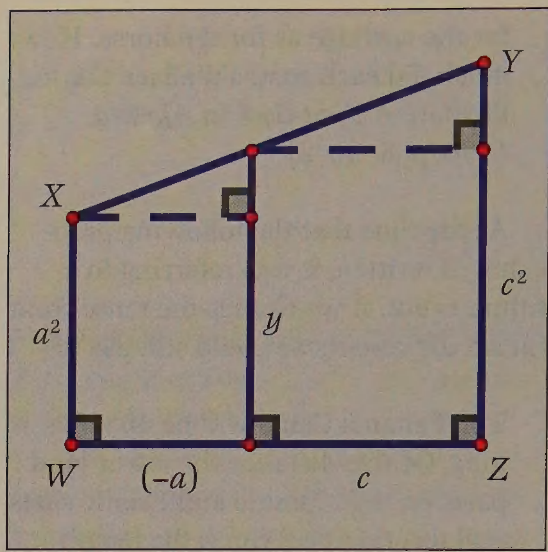


Fig. 2 (Sweeney)

Figure 2 (Sweeney) suggests another way to express the area of trapezoid $WXYZ$ in terms of the area of rectangles and right triangles:

$$A = -a^3 + \frac{1}{2}(-a)(y - a^2) + cy + \frac{1}{2}c(c^2 - y)$$

Equating the two expressions for A, we have the following:

$$\begin{aligned}
&\frac{1}{2}(a^2 + c^2)(-a + c) \\
&\quad = -a^3 + \frac{1}{2}(-a)(y - a^2) + cy + \frac{1}{2}c(c^2 - y) \\
&\quad = -a^3 - \frac{1}{2}ay + \frac{1}{2}a^3 + cy + \frac{1}{2}c^3 - \frac{1}{2}cy \\
&\quad = -\frac{1}{2}a^3 + \frac{1}{2}c^3 - \frac{1}{2}ay + \frac{1}{2}cy
\end{aligned}$$

Doubling both sides and factoring the sum of cubes, we have

$$\begin{aligned}
&(a^2 + c^2)(-a + c) \\
&\quad = -a^3 + c^3 + (-a + c)y \\
&\quad = (-a + c)(a^2 + ac + c^2) + (-a + c)y.
\end{aligned}$$

Dividing by $(-a + c)$ gives $a^2 + c^2 = a^2 + ac + c^2$. So $y = -ac$.

Michael Sweeney
sweeneym@mercersburg.edu
Mercersburg Academy
Mercersburg, PA, Feb. 2, 2014

PROBLEM 3, FEBRUARY 2014 CALENDAR

Problem 3 in the February 2014 Calendar (*MT* Feb. 2014, vol. 107, no. 6, p. 440) was stated as follows:

> The numbers 1–9 are written on nine index cards, one number per card. Arrange the cards into three piles so that the sum of the numbers in each pile is 15.

Thinking outside the Box

There is an additional solution to problem 3 in the February 2014 calendar: 9, 6; 8, 7; and 1, 2, 3, 4, 5. The problem does not state that the three piles need to contain three cards each although, according to the discussion, that is what the authors were thinking.

Karl Singer
singer7@comcast.net
Kensington, NH, Feb. 2, 2014

Thinking outside the Box, Too

As a high school mathematics teacher, I like to post *MT* Calendar problems in my classroom as extra-credit opportunities for my students. One that I posted was problem 3 from the February 2014 Calendar:

I came to the solutions that were published: {2, 4, 9; 1, 6, 8; 3, 5, 7} and {1, 5, 9; 3, 4, 8; 2, 6, 7}. My students, however, came up with an additional solution: {1, 2, 3, 4, 5; 6, 9; 7, 8}. They astutely pointed out that nowhere in the problem did it state that the piles must have an equal number of cards.

Additional unique solutions began to appear. In addition to the solutions indicated above, we found the following:

{1, 2, 4, 8; 3, 5, 7; 6, 9}
{1, 2, 3, 9; 4, 5, 6; 7, 8}
{1, 2, 5, 7; 3, 4, 8; 6, 9}
{1, 3, 5, 6; 2, 4, 9; 7, 8}
{1, 3, 4, 7; 2, 5, 8; 6, 9}
{2, 3, 4, 6; 1, 5, 9; 7, 8}

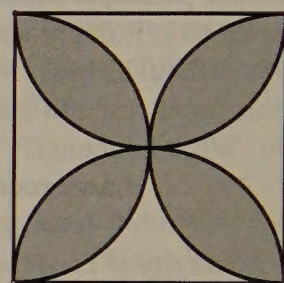
Kudos to my students for thinking outside the box and not making false assumptions!

Jordan Brooks
jbrooks@connectionseducation.com
Nexus Academy of Royal Oak
Beverly Hills, MI, Feb. 13, 2014

PROBLEM 12, FEBRUARY 2014 CALENDAR

Problem 12 of the February 2014 Calendar (*MT* Feb. 2014, vol. 107, no. 6, p. 440), was stated as follows:

> Four semicircles are drawn in the interior of a square using each side of the square as a diameter. If the area of the square is 64, find the area of the shaded region.



As an alternate solution, consider a simplified figure (see **fig. 1 [DeCesare]**). Since the area of the square is 64, the side must be 8. The two shaded regions taken together form the interior of a circle of radius 4; thus, the area of the circle is 16π. Hence, the area of the unshaded region is $64 - 16\pi$. In the original diagram, we can see that we have another unshaded region of area $64 - 16\pi$. Thus,

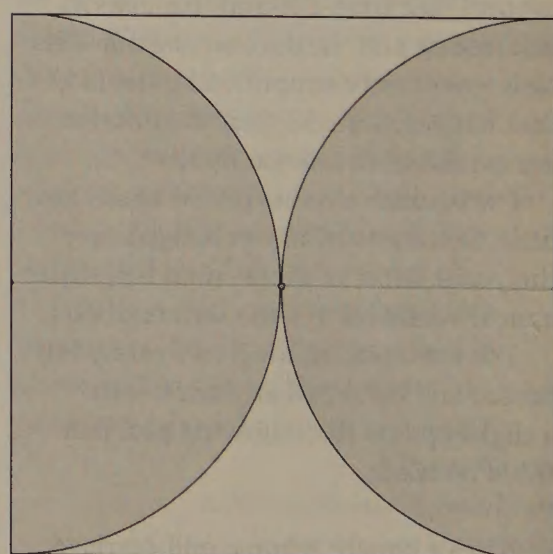


Fig. 1 (DeCesare)

for the original diagram, the area of the shaded region is 64 – (area of unshaded regions) = 64 – (128 – 32π), or 32π – 64 square units.

Rich DeCesare
decesarer1@southernct.edu
Southern Connecticut State University
New Haven, CT, Feb. 16, 2014

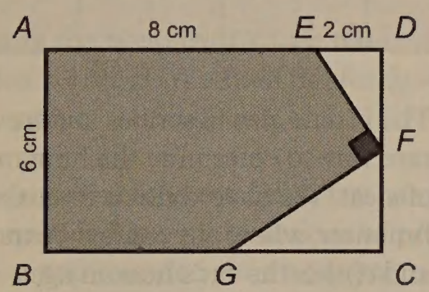
ANOTHER MIDDLE SCHOOL PROBLEM BECOMES AN ALGEBRA PROBLEM

The Palette of Problems in the September 2013 issue of *Mathematics Teaching in the Middle School* contains another excellent problem suitable for beginning algebra students, no matter what grade.

Problem 10, Palette of Problems, MTMS

Problem 10 (*MTMS*, vol. 19, no. 2, p. 83) states:

> In rectangle *ABCD*, *F* is the midpoint of segment *DC*. What is the area of pentagon *AEFGB*?



Suppose that segment *AE* has length 8 cm and segment *AB* has length 6 cm but that segment *BG* has specified length 6 cm and *F* is some point (not necessarily the midpoint) between points *C* and point *D* such that angle *GFE* is a right angle. We can accomplish this by sliding any plane figure with a right angle corner, so that the corner is on side *CD* at *F*. Points *E* and *G* of the rectangle are contained in the two adjacent sides of the right-angled figure. So we visually expect two sets of right triangles *EDF* and *FCG* to appear in our algebraic solution for finding the area of *AEFGB*.

Let x be the length of segment *DF*; then the length of segment *CF* is $6 - x$. We also know that the length of segment *GC* is 4. Since angle *GFE* is a right angle, then angle *GFC* and angle *DFE* are complementary. Hence, angles *GFC* and *FED* are congruent, and triangles *GCF* and *FDE* are similar. The corresponding sides will form a proportion:

$$2/(6 - x) = x/4 \rightarrow x^2 - 6x + 8 = 0$$

So $(x - 4)(x - 2) = 0 \rightarrow x = 4$ or $x = 2$.

If $x = 4$, then $6 - x = 2$, and the area of triangle *EDF* is $(1/2)(2)(4)$ cm^2 = 4 cm^2, and the area of *FCG* is $(1/2)(2) \cdot (4)$ = 4 cm^2. The area of *AEFGB* = $[(10)(6) - 4 - 4]$ cm^2 = 52 cm^2.

If $x = 2$, then $6 - x = 4$. Then the area of *EDF* = $(1/2)(2)(2)$ cm^2 = 2 cm^2, and the area of *FCG* = $(1/2)(4)(4)$ cm^2 = 8 cm^2. Hence, the area of *AEFGB* = $(60 - 2 - 8)$ cm^2 = 50 cm^2.

David Call
dcall@roadrunner.com
Fort Ann Central School (retired)
Fort Ann, NY, Sept. 1, 2013

A MAGIC SQUARE FOR 2014

A *magic square* is a square array of different numbers, with the property that the sum of every row, column, and diagonal is the same number, called the *magic sum*.

A typical magic square is an arrangement of the natural numbers 1, 2, 3, … n^2. It can be shown that the magic sum N of an $n \times n$ normal magic square is $N = n(n^2 + 1)/2$.

For instance, a 4 × 4 normal magic square has a magic sum of $N = 4(4^2 + 1)/2 = 34$. One such 4 × 4 normal magic square is shown in **figure 1 (Sriskandarajah)**.

8	11	14	1
13	2	7	12
3	16	9	6
10	5	4	15

Fig. 1 (Sriskandarajah)

503	506	509	496
508	497	502	507
498	511	504	501
505	500	499	510

Fig. 2 (Sriskandarajah)

Adding 495 to every entry in this square produces a magic square with the magic sum 2014 because $(2014 - N)/n = (2014 - 34)/4 = 495$. (See **fig. 2 [Sriskandarajah]**.)

Notice that rotating or reflecting the array will produce another magic square.

Jeganathan Sriskandarajah
jsriskandara@madisoncollege.edu
Madison College
Madison, WI, Sept. 22, 2013

AN EASIER WAY FOR STUDENTS TO SEE "SOHCAHTOA"

As a second-semester senior at Towson University, I was student teaching at Owings Mills High School in Owings Mills, Maryland, presenting an introductory lesson on trigonometric ratios. The objective of the lesson was for students to be able to find the sine, cosine, and tangent of an angle. I was showing students the mnemonic "SOHCAHTOA," which we all know is a great device to help remember how to find the sine (opposite/hypotenuse), cosine (adjacent/hypotenuse), and tangent (opposite/adjacent).

As I was writing this mnemonic on the board to explain it to the students, it dawned on me to write it as in **figure 1 (Reilly)**, rather than just write it out horizontally. The students seemed to really understand the mnemonic this way, much better than my earlier class had when I just wrote out "SOHCAHTOA." I think this approach can help students see trigonometric ratios more easily.

Brendan Reilly
breill1@students.towson.edu
Student, Towson University
Towson, MD, Mar. 4, 2014

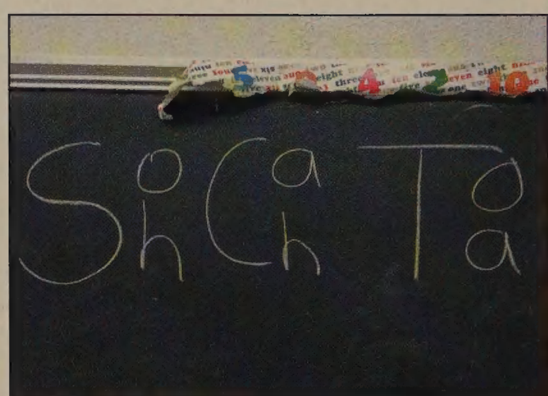


Fig. 1 (Reilly)

Michael A. Jones and Jennifer Wilson

Cats and Dogs: Calculating Human Age

Your Dog's Age	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Age in Human Years	0	12	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88

Source: Gayle Hickman, "How Old Is My Pet? Correctly Calculate Your Dog or Cat's Age!" March 2, 2012, http://shine.yahoo.com/pets/old-pet-correctly-calculate-dog-cats-age-162200633.html

Table 1 ("Cats and Dogs: Calculating Human Age")

Your Cat's Age	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Age in Human Years																			

How can we determine the "human age" of a cat or dog? We used to rely on a simple formula: If the cat or dog is n years old, then its human age is $7n$ years. This folk formula has been updated to be more accurate and uses different formulas for dogs and cats. (See Gayle Hickman, "How Old Is My Pet? Correctly Calculate Your Dog or Cat's Age!" March 2, 2012, http://shine.yahoo.com/pets/old-pet-correctly-calculate-dog-cats-age-162200633.html.)

Media Clips appears in every issue of *Mathematics Teacher*, offering readers contemporary, authentic applications of quantitative reasoning based on print or electronic media. All submissions should be sent to the editors. For information on the department and guidelines for submitting a clip, visit http://www.nctm.org/publications/content.aspx?id=10440#media.

Department editors

Louis Lim, *louis.lim1@gmail.com, Bayview Secondary School, Richmond Hill, ON, Canada; and* ***Chris Bolognese***, *cbolognese@uaschools.org, Upper Arlington City Schools, Columbus, OH*

The table showing a dog's age was given in the original article by Hickman. Use it to answer questions 1 through 3.

1. Use words to explain how to compute a dog's human age. Pay particular attention to the values when a dog is between 0 and 2 years old and more than 2 years old.

2. What is the human age of a puppy that is 6 months old? What is the human age of a dog that is 8.5 years old?

3. Generalize your answer to question 2 to write a function where n represents how old the dog is and $d(n)$ is the dog's human age. (If this problem is too difficult, try adjusting the function $c(n)$ below for cats.)

4. At what time or times in its life, n years old, will a dog be the same human age calculated under both the old and the new methods? What would be its human age?

The article also describes a more accurate way to determine the human age of a cat. We have written the rule as a function, where the cat is n years old and $c(n)$ is the cat's human age:

$$c(n) = \begin{cases} 15n & \text{if } 0 \le n \le 1 \\ 15 + 9(n-1) & \text{if } 1 \le n \le 2 \\ 24 + 4(n-2) & \text{if } 2 \le n \end{cases}$$

5. Use the equation from question 4 to fill in **table 1 ("Cats and Dogs: Calculating Human Age")**.

6. What is the human age of a kitten that is 6 months old? What is the human age of a cat that is 8.5 years old?

7. Explain why a cat and a dog are the same human age after they are 2 years old.

8. At what time or times in its life, at n years old, will a cat be the same human age calculated under both the old and new methods? What would be its human age? (Hint: Think about your answers to questions 4 and 7.)

Rounding at the Register

The Price Rounding Act of 1989 was introduced in the U.S. House of Representatives by Jim Kolbe (R-Ariz.) to eliminate the penny in cash transactions by rounding to the nearest nickel. Although this and adapted proposals from 2001 and 2006 were never voted on, this practice was recently implemented in select Chipotle restaurants in busy urban areas (Boston, northern New Jersey, and New York City) in an effort to speed up transactions. (See Karin Price Mueller, "Bamboozled: Chipotle Receipts Finally Start Making Sense," *Newark* [*N.J.*] *Star-Ledger*, August 27, 2012, http://www.nj.com/business/index.ssf/2012/08/bamboozled_chipotle_receipts_f.html.)

Answer the following questions to explore the consequences of rounding food charges, food and tax total, and final bills to the nearest nickel.

1. Using the tax rate of 7% on food, what would be the final bill for food charges of $4.50, $8.52, and $12.86?

2. Because the food total plus tax has to be rounded to avoid fractions of pennies, you rounded your answers to question 1. How did you round to the nearest penny in your answers for question 1?

3. Compute the final bills for the food and tax totals for question 1 under the "nearest nickel" rounding rule.

4. Given the amounts in question 1, can you find two items for which it is more expensive to buy the items together (because of rounding to the nearest nickel) than it is to buy them separately?

The article goes on to recount an experience at a local New Jersey Chipotle restaurant: "Bamboozled made two purchases . . . and our purchases were added correctly, with totals of $8.00 and $7.50. We noticed the prices in that store were odd, but we figured that was so that when the tax was added, the final bill would end in a zero or a five to avoid the dreaded pennies." Perhaps Bamboozled went to the Chipotle restaurant in Clifton, New Jersey. The unusual prices for certain items from the Clifton Chipotle menu are listed in **table 1 ("Rounding at the Register")**.

Table 1 ("Rounding at the Register")

	Steak Burrito	Vegetarian Burrito	Chips and Guacamole	Small Soda
Price (in $) without tax	7.01	6.55	3.28	1.87
Calculated bill (in $) including tax				
Final bill (in $) with penny rounding				

5. Fill in **table 1 ("Rounding at the Register")** by computing the bills with 7% tax and rounding to the nearest penny.

6. What rounding strategies could be used so that the final bills for items in **table 1 ("Rounding at the Register")** give round numbers (i.e., those that end in 0 or 5)?

7. If a restaurant in New Jersey wanted the price of an item to be $10 after the 7% tax, what should the pretax price be? Is the answer different under "nearest penny" and "nearest nickel" rounding policies?

MEDIA CLIPS solutions

Table 2 ("Cats and Dogs: Calculating Human Age")

Your Cat's Age	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
Age in Human Years	0	15	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	84	88

"Cats and Dogs: Calculating Human Age" answers

1. For the first two years of its life, a dog ages as if each year were 12 human years. For each additional year that the dog lives, the dog ages as if each year were 4 human years.

2. According to the new model, a dog that is 6 months old has a human age of 6 years. A dog that is 8.5 years old has a human age of 50.

3. Let $d(n)$ be human age of a dog if it has lived n years. Then,

$$d(n) = \begin{cases} 12n & \text{if } 0 \le n \le 2 \\ 24 + 4(n-2) & \text{if } 2 \le n \end{cases}.$$

4. The old method is described by the function $o(n) = 7n$. The question can be restated in this way: For which values of n do $d(n)$ and $o(n)$ agree? Notice that $n = 0$ is an obvious answer. But the two functions agree for one other value of n because $d(n)$ increases more quickly than $o(n)$ initially but less quickly after $n = 2$. The second intersection occurs when $o(n) = 7n = d(n) = 24 + 4(n - 2)$ or $n = 16/3$. After 5 1/3 years, a dog's age is equivalent to 37 1/3 human years, according to both formulas. These intersection points can also be determined graphically (see **fig. 1 ["Cats and Dogs: Calculating Human Age"]**.

5. The nineteen ages can also be found in the article (see **table 2 ["Cats and Dogs: Calculating Human Age"]**).

6. A cat that is 6 months old would have a human age of 7.5 years. A cat that is

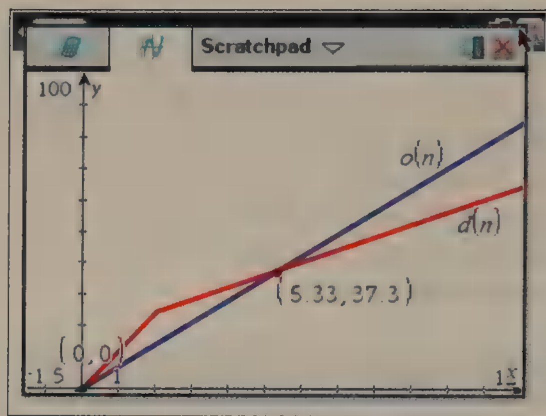


Fig. 1 ("Cats and Dogs: Calculating Human Age")

8.5 years old has a human age of 50.

7. After a cat and a dog are 2 years old, then their human age increases by 4 for every year of life. They are the same human age at all times after 2 years old because they both have a human age of 24 when they are 2 years old. Another way to state this relationship is that both linear functions pass through the same point and have the same slope; thus, they are equivalent functions for $n \ge 2$.

8. In light of the answer to question 7, the answer must be the same as the answer to question 4. Hence, after 5 1/3 years, a cat's age is equivalent to 37 1/3 human years, according to both formulas.

"Rounding at the Register" answers

1. For food charges of \$4.50, \$8.52, and \$12.86, the bills are \$4.82, \$9.12, and \$13.76, respectively.

2. Before rounding, the totals from question 1 were \$4.815, \$9.1164, and \$13.7602, respectively (see **table 2 ["Rounding at the Register"]**). These amounts were rounded to the nearest penny (5/10 of a penny and above rounded up; less than 5/10 of a penny rounded down).

3. If the "nearest nickel" rounding rule was used, then the final bills for food charges of \$4.50, \$8.52, and \$12.86 would be \$4.80, \$9.10, and \$13.75.

4. If the food charges of \$4.50 and \$8.52 are added together to get \$13.02, then the posttax amount of \$13.9314 is rounded to \$13.95 under the "nearest nickel" rounding rule. From the answer to question 3, we see that the sum of the bills being paid separately is \$4.80 + \$9.10 = \$13.90. Thus, buying the items that cost \$4.50 and \$8.52 separately would be cheaper than if purchased together under the "nearest nickel" rule.

5. See **table 3 ("Rounding at the Register")**.

6. The prices of the vegetarian burrito and chips and guacamole would round to end in 0 or 5 under both the "rounding down" and the "nearest nickel" policies.

7. Under the "nearest penny" rounding rules, the pretax price should be \$9.35. To find this, solve \$10.00/1.07 to get \$9.3457… and round to \$9.35. Verify that \$9.35 • 1.07 = \$10.0045 is rounded to \$10.00.

   Under a "nearest nickel" rounding policy, any calculated bill between \$9.96 and \$10.04 (exclusive) could

round (up or down) to exactly \$10.00. These bills could have been generated by possible prices of \$9.3084… and \$9.3831…. Thus, items in a price range of \$9.31 to \$9.38 would have final bills rounded to exactly \$10.00 under a nickel-rounding policy.

MICHAEL A. JONES, maj@ams.org, is an editor at Mathematical Reviews in Ann Arbor, Michigan. He regularly teaches in the Michigan Math and Science Scholars summer program for high school students at the University of Michigan. **JENNIFER WILSON,** wilsonj@newschool.edu, teaches at Eugene Lang College, New School for Liberal Arts, in New York City. She is interested in the visualization of mathematical and financial concepts and the geometry of pop-up books.

Table 2 ("Rounding at the Register")

Food Charge (in \$)	Calculated Food and Tax Total (in \$)	Final Bill with Nearest Penny Rounding	Final Bill with Nearest Nickel Rounding
4.50	4.815	4.82	4.80
8.52	9.1164	9.12	9.10
12.86	13.7602	13.76	13.75

Table 3 ("Rounding at the Register")

	Steak Burrito	Vegetarian Burrito	Chips and Guacamole	Small Soda
Price (in \$) without tax	7.01	6.55	3.28	1.87
Calculated bill (in \$) including tax	7.5007	7.0085	3.5096	2.0009
Final bill (in \$) with penny rounding	7.50	7.01	3.51	2.00

Günhan Caglayan and Horace Buddoo

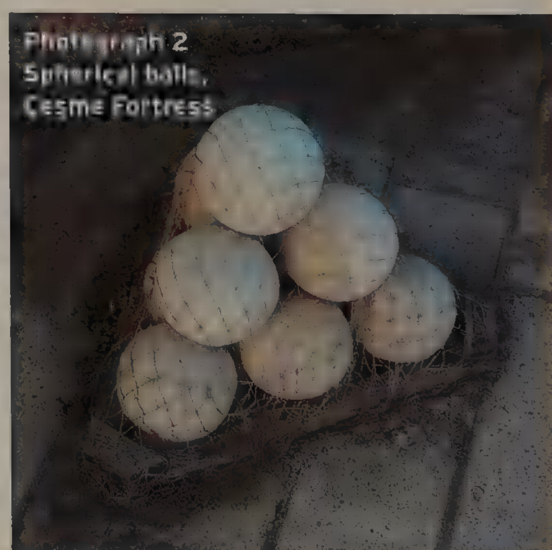
Tetrahedral Numbers

Photograph 1
Spherical balls, Çeşme Fortress, Çeşme, Turkey

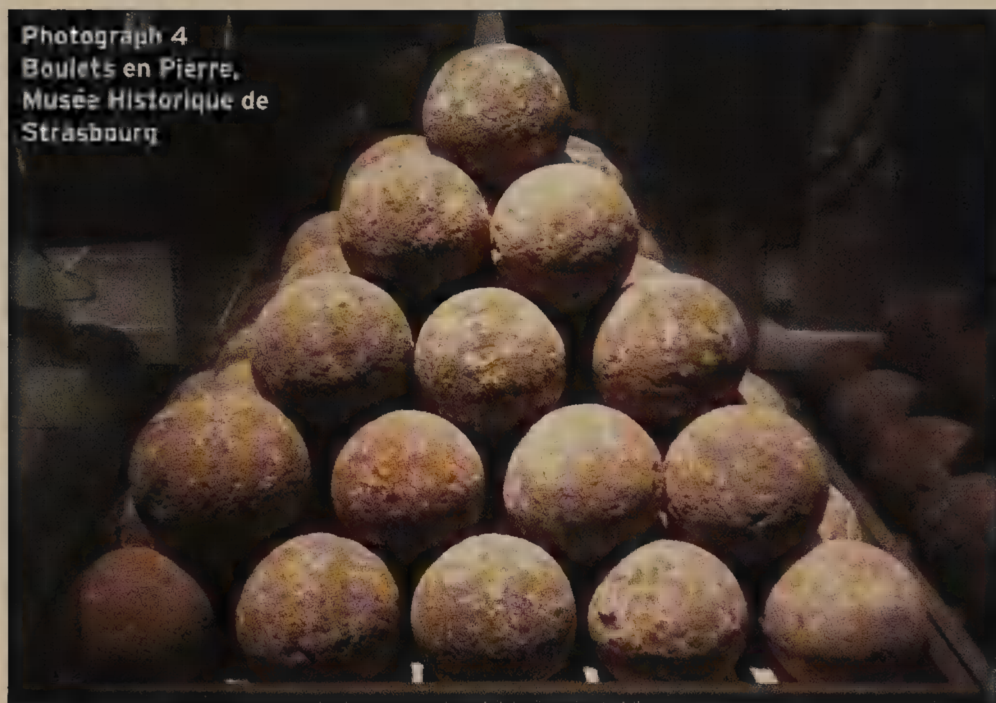
Mathematical Lens uses photographs as a springboard for mathematical inquiry and appears in every issue of *Mathematics Teacher*. All submissions should be sent to the department editors. For more background information on Mathematical Lens and guidelines for submitting a photograph and questions, please visit http://www.nctm.org/publications/content.aspx?id=10440#lens.

Department editors
Ron Lancaster, *ron2718@nas.net, University of Toronto, ON, Canada, and* ***Brigitte Bentele,*** *brigitte.bentele@trinityschoolnyc.org, Trinity School, New York, NY*

A *tetrahedral number* corresponds to a three-dimensional figure in the shape of a tetrahedron made of spheres. Each tetrahedral number can be expressed as the sum of consecutive triangular numbers: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, . . . The nth tetrahedral number is defined as the sum of the first n triangular numbers. For example, the third tetrahedral number, 10, is the sum of the first three triangular numbers: 1, 3, 6 (see **photographs 1** and **2**). The sequence of tetrahedral numbers is $\{1, 4, 10, 20, 35, 56, 84, 120, 165, 220, \ldots\}$.



Photograph 2
Spherical balls, Çeşme Fortress



1. (*a*) Use the triangular number formula $T_k = (1/2)k(k+1)$ to obtain the formula for the nth tetrahedral number $\theta_n = (1/6)n(n+1) \cdot (n+2)$.

   (*b*) Show that each tetrahedral number corresponds to a binomial coefficient. Also describe the location of the tetrahedral numbers in Pascal's triangle.

   (*c*) Tetrahedral numbers have other interesting properties as well. Prove that the infinite series $\sum_{k=1}^{\infty}(1/\theta_k)$ is convergent and determine the sum of this series.

   (*d*) Use a spreadsheet to help you determine which numbers are tetrahedral and which triangular.

2. (*a*) A square pyramidal number corresponds to a three-dimensional figure in the shape of a square-based pyramid. The nth square pyramidal number is defined as the sum of the first n square numbers. The arrangement shown in **photographs 3** and **4** represents the fifth square pyramidal number, 55, which is the sum of the first five square numbers: 1, 4, 9, 16, 25. The sequence of square pyramidal numbers is $\{1, 5, 14, 30, 55, 91, 140, 204, 285, 385, \ldots\}$. Show that each square pyramidal number P_n can be written as the sum of two consecutive tetrahedral numbers.

   (*b*) **Figure 1** is obtained by reflecting the pyramid in **photograph 4** across the "square base" using The Geometer's Sketchpad®. The resulting arrangement represents the fifth octahedral number, 85, which is the sum of the fourth (below) and the fifth (above) square pyramidal numbers: 30 and 55. Derive a formula for the nth octahedral number, O_n. Express this formula compactly as a product.

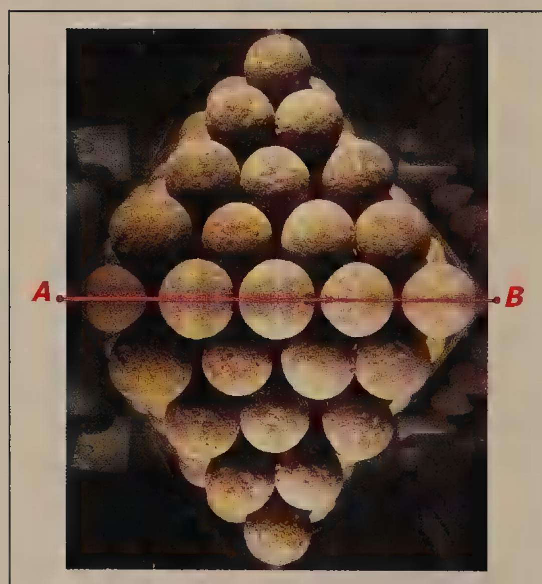


Fig. 1 An octahedron is formed through reflection of a square-based pyramid.

3. Prove the following relations relating triangular (T_n), tetrahedral (θ_n), square pyramid (P_n), and octahedral numbers (O_n).

   (*a*) $O_n + 4\theta_{n-1} = \theta_{2n-1}$

   (*b*) $\theta_n + 2\theta_{n-1} + \theta_{n-2} = O_n$

   (*c*) $O_n + 2\theta_{n-1} = n^3$

   (*d*) $nT_n - \theta_{n-1} = P_n$

MATHEMATICAL LENS solutions

1. (*a*) Using the sum of counting numbers formula, $\sum_{k=1}^{n} k = (1/2)n(n+1)$, and the sum of squares formula, $\sum_{k=1}^{n} k^2 = (1/6)n(n+1)(2n+1)$, we have the following:

$$\begin{aligned}
\theta_n &= \sum_{k=1}^{n} T_k = \sum_{k=1}^{n} \frac{1}{2}k(k+1) \\
&= \frac{1}{2}\sum_{k=1}^{n}(k^2+k) = \frac{1}{2}\left(\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k\right) \\
&= \frac{1}{2}\left(\frac{1}{6}n(n+1)(2n+1) + \frac{1}{2}n(n+1)\right) \\
&= \frac{1}{12}n(n+1)\left[(2n+1)+(3)\right] \\
&= \frac{1}{12}n(n+1)(2n+4) \\
&= \frac{1}{6}n(n+1)(n+2)
\end{aligned}$$

(*b*) Because $\theta_n = (1/6)n(n+1)(n+2)$ can be written as

$$\theta_n = \frac{(n+2)(n+1)n}{3 \cdot 2 \cdot 1},$$

we see that each tetrahedral number corresponds to a binomial coefficient. In particular, the *n*th tetrahedral number corresponds to

$$\binom{n+2}{3}.$$

In Pascal's triangle, these coefficients are located in the fourth "diagonal" from right or left (see **fig. 2**).

(*c*) Because

$$\frac{1}{\theta_k} = \frac{6}{k(k+1)(k+2)}$$

is less than $6/k^3$ for all k and because $\sum_{k=1}^{\infty}(6/k^3)$ is a convergent series by the *p*-series test, the series $\sum_{k=1}^{\infty}(1/\theta_k)$ is convergent by the comparison test. To determine the value of this series, we use the fractional decomposition technique. Let

$$S_n = \sum_{k=1}^{n}\frac{1}{\theta_k} = \sum_{k=1}^{n}\frac{6}{k(k+1)(k+2)}.$$

To obtain a fractional decomposition, we set

$$\frac{6}{k(k+1)(k+2)} = \frac{A}{k} + \frac{B}{k+1} + \frac{C}{k+2}.$$

Equivalently, we obtain

$$\begin{aligned}
&A(k+1)(k+2) + Bk(k+2) + Ck(k+1) = 6 \rightarrow \\
&Ak^2 + 3Ak + 2A + Bk^2 + 2Bk + Ck^2 + Ck = 6.
\end{aligned}$$

Solving the system of three equations $(A+B)k^2 = 0$, $(3A+2B+C) \cdot k = 0$, and $2A = 6$ yields $A = C = 3$, $B = -6$. The original expression can thus be written as

$$\begin{aligned}
S_n &= \sum_{k=1}^{n}\left(\frac{3}{k} - \frac{6}{k+1} + \frac{3}{k+2}\right) \\
&= 3\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{2}{k+1} + \frac{1}{k+2}\right) \\
&= 3\left[\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) - \sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right)\right].
\end{aligned}$$

We can write H_n denoting the *n*th harmonic number $H_n = \sum_{k=1}^{n}(1/k)$. The expression $\sum_{k=1}^{n}(1/(k+1))$ can be written in terms of H_n, as follows:

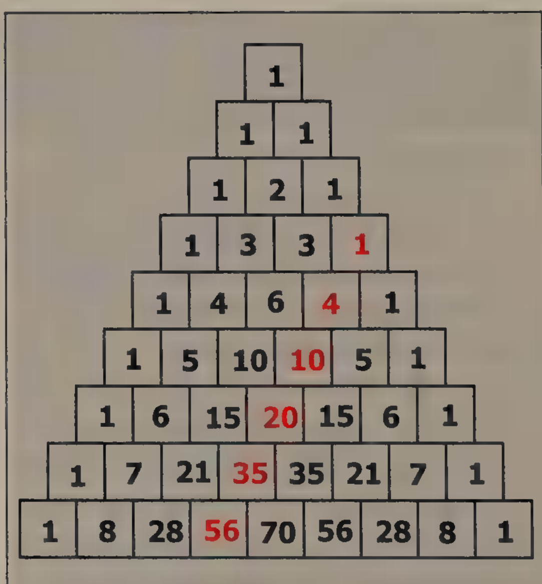


Fig. 2 Tetrahedral numbers are binomial coefficients listed in a diagonal of Pascal's triangle.

$$\begin{aligned}
\sum_{k=1}^{n}\frac{1}{k+1} &= -1 + \frac{1}{n+1} + \sum_{k=1}^{n}\frac{1}{k} \\
&= -1 + \frac{1}{n+1} + H_n.
\end{aligned}$$

Substitution yields

$$\begin{aligned}
\sum_{k=1}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) &= \sum_{k=1}^{n}\frac{1}{k} - \sum_{k=1}^{n}\frac{1}{k+1} \\
&= H_n - \left(-1 + \frac{1}{n+1} + H_n\right) \\
&= 1 - \frac{1}{n+1}.
\end{aligned}$$

Using a similar approach, we obtain

$$\begin{aligned}
\sum_{k=1}^{n}\left(\frac{1}{k+1} - \frac{1}{k+2}\right) &= \sum_{k=1}^{n}\frac{1}{k+1} - \sum_{k=1}^{n}\frac{1}{k+2} \\
&= \left(-1 + \frac{1}{n+1} + H_n\right) \\
&\quad - \left(-1 - \frac{1}{2} + \frac{1}{n+1} + \frac{1}{n+2} + H\right. \\
&= \frac{1}{2} - \frac{1}{n+2}.
\end{aligned}$$

Substituting these results in the original expression for S_n yields

$$S_n = 3\left(1 - \frac{1}{n+1}\right) - 3\left(\frac{1}{2} - \frac{1}{n+2}\right).$$

The value of the infinite series $\sum_{k=1}^{\infty}(1/\theta_k)$ is therefore

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{1}{\theta_k} &= \lim_{n\to\infty} S_n = 3(1-0) - 3\left(\frac{1}{2} - 0\right) \\
&= 3 - \frac{3}{2} = \frac{3}{2}.
\end{aligned}$$

A visual proof of this fact can be demonstrated with dynamic geometry software, such as GeoGebra (see **fig. 3**). The sequence of blue rectangles above the *x*-axis is obtained through iteration using GeoGebra's spreadsheets. Consecutive points $B_k(k, 3/k)$ form corners of each blue rectangle of width 1 and height equal to the difference of the *y*-coordinates of

the corner points. The sum of the areas of the blue rectangles thus corresponds to the expression

$$3\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right),$$

whereas the sum of the areas of the red rectangles below the x-axis corresponds to the expression

$$3\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right).$$

For a fixed value of n, we match blue-red pairs of equal area, leaving an isolated first blue rectangle and a final red rectangle unmatched. When we calculate the difference

$$3\sum_{k=1}^{n}\left(\frac{1}{k}-\frac{1}{k+1}\right)-3\sum_{k=1}^{n}\left(\frac{1}{k+1}-\frac{1}{k+2}\right),$$

we can eliminate blue-red pairs with a net area of zero. For large values of n, the area of the final red rectangle is small, so the value of the infinite series is therefore the same as the area of the first blue rectangle above the x-axis—namely, 1.5.

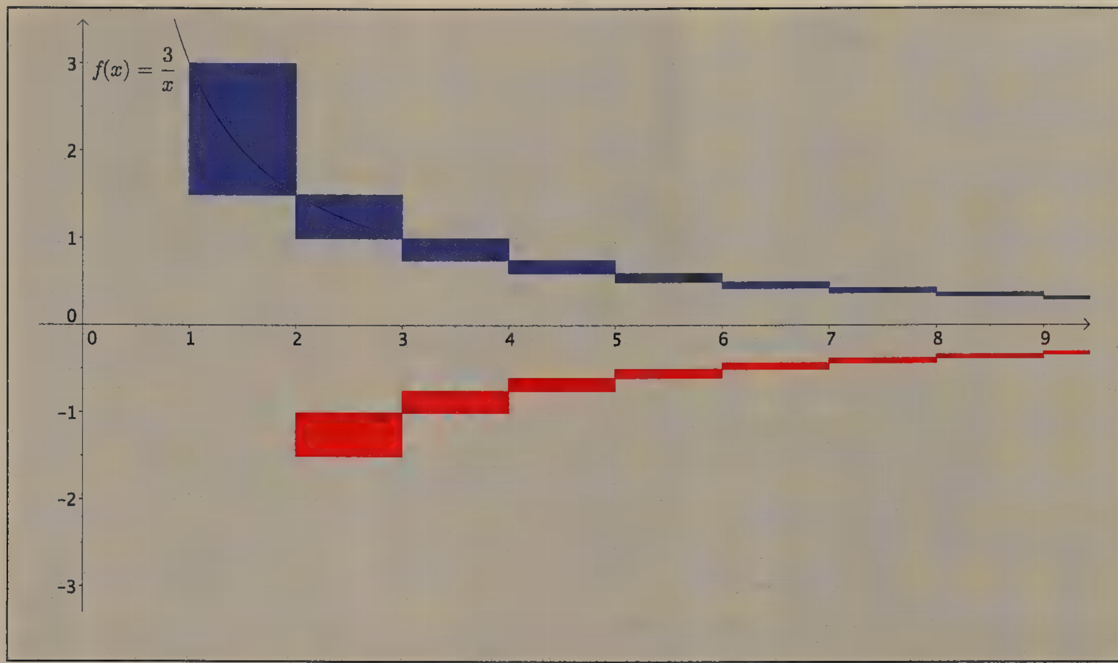


Fig. 3 The net area of rectangles represents the series of reciprocals of tetrahedral numbers.

(*d*) As shown in **figure 4**, numbers that are both triangular and tetrahedral include 1, 10, 120, 1540, and 7140.

2. (*a*) By definition, the nth square pyramidal number can be written as $P_k = \sum^{n}_{k=1} k^2$. The proof follows from the decomposition

$$\begin{aligned} k^2 &= \frac{k^2}{2}+\frac{k^2}{2} \\ &= \left(\frac{k^2}{2}+\frac{k}{2}\right)+\left(\frac{k^2}{2}-\frac{k}{2}\right) \\ &= \frac{1}{2}k(k+1)+\frac{1}{2}k(k-1). \end{aligned}$$

By definition,

$$\sum_{k=1}^{n}\frac{1}{2}k(k+1)=\sum_{k=1}^{n}T_k=\theta_n.$$

Similarly,

$$\begin{aligned} \sum_{k=1}^{n}\frac{1}{2}k(k-1) &= 0+\sum_{k=2}^{n}\frac{1}{2}k(k-1) \\ &= \sum_{j=1}^{n-1}\frac{1}{2}(j+1)(j) \\ &= \sum_{j=1}^{n-1}T_j=\theta_{n-1}. \end{aligned}$$

Substitution yields the desired result:

$$\begin{aligned} P_n &= \sum_{k=1}^{n}k^2=\sum_{k=1}^{n}\left(\frac{1}{2}k(k+1)+\frac{1}{2}k(k-1)\right) \\ &= \sum_{k=1}^{n}\frac{1}{2}k(k+1)+\sum_{k=1}^{n}\frac{1}{2}k(k-1) \\ &= \theta_n+\theta_{n-1} \end{aligned}$$

(*b*) Considering the symmetry with respect to the "square base" of the regular octahedron, the nth octahedral number can be thought of as made of the nth square number (i.e., the base) and a pair of square pyramidal numbers at the $(n - 1)$st level, which translates to the following statements: $O_n = n^2 + 2P_{n-1}$. In answer 2(*a*), we showed that $P_n = \theta_n + \theta_{n-1}$ or, equivalently, $P_{n-1} = \theta_{n-1} + \theta_{n-2}$. Therefore, $O_n = n^2 + 2P_{n-1} = n^2 + 2(\theta_{n-1} + \theta_{n-2})$. Substituting, factoring, and simplifying yield

$$\begin{aligned} O_n &= n^2+2\left(\frac{(n-1)n(n+1)}{6}+\frac{(n-2)(n-1)n}{6}\right) \\ &= n^2+\frac{1}{3}n(n-1)\left[(n+1)+(n-2)\right] \\ &= n^2+\frac{1}{3}n(n-1)(2n-1) \\ &= n\left[n+\frac{1}{3}(n-1)(2n-1)\right] \\ &= \frac{n}{3}\left(3n+2n^2-3n+1\right) \\ &= \frac{n(2n^2+1)}{3}. \end{aligned}$$

This result also suggests that for any n, either n or $(2n^2 + 1)$ must be divisible by 3.

3. (*a*) The identity $O_n + 4\theta_{n-1} = \theta_{2n-1}$ can be visualized by sticking four of the same tetrahedral numbers at the $(n - 1)$st level (made of spherical balls, for instance) onto the bottom faces of an octahedral number at the nth level. The outcome is another tetrahedral number.

Algebraically, we have the following:

A	B	C	A	B	C	A	B	C	A	B	C	A	B	C
1	1	1	31	496	5456	61	1891	39711	91	4186	129766	121	7381	302621
2	3	4	32	528	5984	62	1953	41664	92	4278	134044	122	7503	310124
3	6	10	33	561	6545	63	2016	43680	93	4371	138415	123	7626	317750
4	10	20	34	595	7140	64	2080	45760	94	4465	142880	124	7750	325500
5	15	35	35	630	7770	65	2145	47905	95	4560	147440	125	7875	333375
6	21	56	36	666	8436	66	2211	50116	96	4656	152096	126	8001	341376
7	28	84	37	703	9139	67	2278	52394	97	4753	156849	127	8128	349504
8	36	120	38	741	9880	68	2346	54740	98	4851	161700	128	8256	357760
9	45	165	39	780	10660	69	2415	57155	99	4950	166650	129	8385	366145
10	55	220	40	820	11480	70	2485	59640	100	5050	171700	130	8515	374660
11	66	286	41	861	12341	71	2556	62196	101	5151	176851	131	8646	383306
12	78	364	42	903	13244	72	2628	64824	102	5253	182104	132	8778	392084
13	91	455	43	946	14190	73	2701	67525	103	5356	187460	133	8911	400995
14	105	560	44	990	15180	74	2775	70300	104	5460	192920	134	9045	410040
15	120	680	45	1035	16215	75	2850	73150	105	5565	198485	135	9180	419220
16	136	816	46	1081	17296	76	2926	76076	106	5671	204156	136	9316	428536
17	153	969	47	1128	18424	77	3003	79079	107	5778	209934	137	9453	437989
18	171	1140	48	1176	19600	78	3081	82160	108	5886	215820	138	9591	447580
19	190	1330	49	1225	20825	79	3160	85320	109	5995	221815	139	9730	457310
20	210	1540	50	1275	22100	80	3240	88560	110	6105	227920	140	9870	467180
21	231	1771	51	1326	23426	81	3321	91881	111	6216	234136	141	10011	477191
22	253	2024	52	1378	24804	82	3403	95284	112	6328	240464	142	10153	487344
23	276	2300	53	1431	26235	83	3486	98770	113	6441	246905	143	10296	497640
24	300	2600	54	1485	27720	84	3570	102340	114	6555	253460	144	10440	508080
25	325	2925	55	1540	29260	85	3655	105995	115	6670	260130	145	10585	518665
26	351	3276	56	1596	30856	86	3741	109736	116	6786	266916	146	10731	529396
27	378	3654	57	1653	32509	87	3828	113564	117	6903	273819	147	10878	540274
28	406	4060	58	1711	34220	88	3916	117480	118	7021	280840	148	11026	551300
29	435	4495	59	1770	35990	89	4005	121485	119	7140	287980	149	11175	562475
30	465	4960	60	1830	37820	90	4095	125580	120	7260	295240	150	11325	573800

Fig. 4 Some triangular numbers (column B) are also tetrahedral (column C).

$$
\begin{aligned}
O_n + 4\theta_{n-1} &= \frac{n(2n^2+1)}{3} + 4\,\frac{(n-1)n(n+1)}{6} \\
&= \frac{n(2n^2+1)}{3} + \frac{2(n-1)n(n+1)}{3} \\
&= \frac{n}{3}\left[2n^2+1+2(n-1)(n+1)\right] \\
&= \frac{n}{3}(2n^2+1+2n^2-2) \\
&= \frac{n}{3}(4n^2-1) \\
&= \frac{n}{3}(2n-1)(2n+1) \\
&= \frac{2}{2}\bullet\frac{n}{3}(2n-1)(2n+1) \\
&= \frac{(2n-1)(2n)(2n+1)}{6} \\
&= \theta_{2n-1}
\end{aligned}
$$

How big is the "new" tetrahedral number? The ratio of these numbers is

$$
\begin{aligned}
\frac{\theta_{2n-1}}{\theta_{n-1}} &= \frac{\dfrac{(2n-1)(2n)(2n+1)}{6}}{\dfrac{(n-1)(n)(n+1)}{6}} \\
&= \frac{(2n-1)(2n)(2n+1)}{(n-1)(n)(n+1)} \\
&= \frac{8\left(n-1/2\right)\left(n+1/2\right)}{(n-1)(n+1)}.
\end{aligned}
$$

Taking the limit, we obtain $\lim_{n\to\infty}(\theta_{2n-1}/\theta_{n-1}) = 8$. This result agrees with the geometric situation that it is possible to "convert" a regular octahedron into a regular tetrahedron by adding four congruent tetrahedrons (whose faces are congruent to the faces of the octahedron) to its four bottom faces. The volume of the new tetrahedron will be eight times the volume of the old tetrahedron. Moreover, the ratio of the side lengths (constant of proportionality) is exactly 2.

(*b*) Using answer 2(*a*), we have

$$
\begin{aligned}
&\theta_n + 2\theta_{n-1} + \theta_{n-2} \\
&\quad = (\theta_n + \theta_{n-1}) + (\theta_{n-1} + \theta_{n-2}) = P_n + P_{n-1}.
\end{aligned}
$$

Because $P_n = n^2 + P_{n-1}$, substitution yields

$$P_n + P_{n-1} = n^2 + P_{n-1} + P_{n-1} = n^2 + 2P_{n-1} = O_n$$

by answer 2(*b*).

(c) This identity can be visualized by sticking two of the same tetrahedral numbers at the $(n - 1)$st level onto the opposite faces of an octahedral number at the nth level. The outcome is a (sheared) cube!

Algebraically, we have the following:

$$\begin{aligned} O_n + 2\theta_{n-1} &= \frac{n(2n^2+1)}{3} + 2\frac{(n-1)n(n+1)}{6} \\ &= \frac{n(2n^2+1)}{3} + \frac{(n-1)n(n+1)}{3} \\ &= \frac{n}{3}\left[2n^2+1+(n-1)(n+1)\right] \\ &= \frac{n}{3}(2n^2+1+n^2-1) \\ &= \frac{n}{3}(3n^2) \\ &= n^3 \end{aligned}$$

(d) Algebraically, we have the following:

$$\begin{aligned} nT_n - \theta_{n-1} &= n\frac{n(n+1)}{2} - \frac{(n-1)(n)(n+1)}{6} \\ &= \frac{3n(n)(n+1)}{6} - \frac{(n-1)(n)(n+1)}{6} \\ &= \frac{1}{6}n(n+1)\left[3n-(n-1)\right] \\ &= \frac{1}{6}n(n+1)(3n-n+1) \\ &= \frac{1}{6}n(n+1)(2n+1) \\ &= \frac{1}{6}n(n+1)\left[(n+2)+(n-1)\right] \\ &= \frac{1}{6}n(n+1)(n+2) + \frac{1}{6}n(n+1)(n-1) \\ &= \theta_n + \theta_{n-1} \\ &= P_n \end{aligned}$$

GÜNHAN CAGLAYAN, caglayan@fiu.edu, teaches mathematics at Florida International University in Miami. He is interested in preservice mathematics teacher education and student learning through modeling and visualization. **HORACE BUDDOO,** hbuddoo@mrbuddoo.com, teaches algebra and statistics at West Broward High School in Pembroke Pines, Florida. His goal is to help students, parents, and teachers to see mathematics and mathematics education in a new way using innovative and technology-driven methods.

Pose and Solve Varignon Converse Problems

Students' conjectures relating the midpoints of the sides of a quadrilateral are inspired by using The Geometer's Sketchpad and justified by proofs.

José N. Contreras

For me, posing and solving mathematical problems is one of the most exhilarating intellectual activities. The more challenging the problem, the greater the joy I feel in posing it or solving it. The activity of posing and solving problems can enrich learners' mathematical experiences because it fosters a spirit of inquisitiveness, cultivates their mathematical curiosity, and deepens their views of what it means to do mathematics. To achieve these goals, a mathematical problem needs to be at the appropriate level of difficulty, challenging but not insurmountable, and rich enough to be a fruitful source of new and interesting problems.

One such example is based on the classic theorem known as Varignon's theorem: The midpoints E, F, G, and H of the consecutive sides of a quadrilateral $ABCD$ are the vertices of a parallelogram (Oliver 2001a, 2001b). To engage my students in deeper investigations, I reformulated the theorem as an open-ended problem, which I call the Varignon problem (see **fig. 1**). For some of the special quadrilaterals (see **table 1**), we can formulate different converse problems, which I call Varignon converse problems (Contreras 2009).

As learners embark on problem-posing and problem-solving adventures, they need to be equipped with powerful tools of inquiry that will support their efforts in exploring uncharted territory. The first tool that learners need is a systematic problem-posing strategy. Posing converse problems is a strategy that is applicable to a wide range of problems. As Movshovits-Hadar (1988) suggests, most nontrivial mathematical problems are an endless source of surprise. Varignon converse problems, as we will see, are no exception.

The second tool that learners need to continue their investigations is a technological device such as GeoGebra or The Geometer's Sketchpad® that can facilitate the discovery of patterns or regularities to formulate a plausible conjecture (Pólya 1945).

Finally, learners need a mathematical and pedagogical tool to transform their conjecture into a theorem. Such a tool is a proof. Constructing a mathematical proof allows learners not only to verify the validity of a theorem but also, and perhaps more important, to gain insight about *why* it is true. A proof allows the learner to connect the underlying mathematical concepts.

INVESTIGATE THE VARIGNON PROBLEM

A group of prospective secondary school mathematics teachers (referred to as students or class members) who were enrolled in a college geometry course under the direction of the author (referred to as the instructor) used The Geometer's Sketchpad (GSP) to investigate some special cases of the Varignon problem. The class displayed its findings as shown in **table 2** (see Contreras [2009] for details). Because there is not universal agreement about the definitions of the terms *trapezoid*, *isosceles trapezoid*, and *kite*, I have provided the definitions that the class adopted (see **table 3**) and the associated hierarchy of the special quadrilaterals (see **opening art**). By the time the class investigated the Varignon problem, the students had extensive experience in using GSP as well as in constructing proofs applying theories of congruence and similarity of triangles.

In this article, I discuss some of the converse problems and the conjectures and proofs that the students generated. Students supported or refuted their initial conjectures with GSP and then

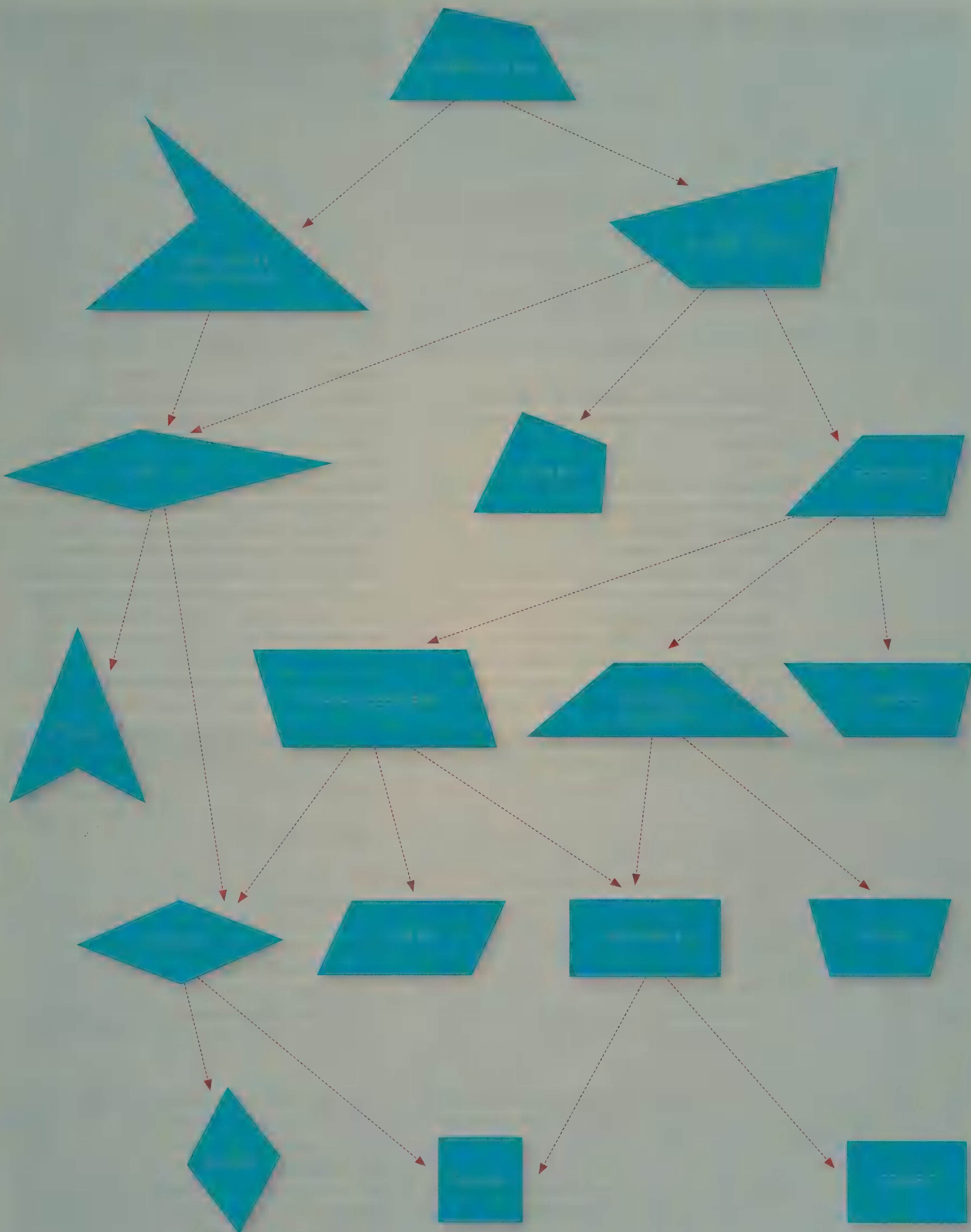


Table 1 Special Varignon Problems

1. Let *ABCD* be a parallelogram and let *E*, *F*, *G*, and *H* be the midpoints of its consecutive sides. Prove that *EFGH* is a parallelogram.
2. *E*, *F*, *G*, and *H* are the midpoints of a rectangle. What type of quadrilateral is *EFGH*?
3. What type of quadrilateral is formed when the midpoints of the sides of a rhombus are joined?
4. Prove that the medial quadrilateral of a square is a square.
5. What type of quadrilateral is formed by [joining] the midpoints of the sides of a trapezoid?
6. If *ABCD* is an isosceles trapezoid and *E*, *F*, *G*, and *H* are the midpoints of its sides, prove that *EFGH* is a rhombus.
7. What type of quadrilateral is the medial quadrilateral of a kite?

Source: Contreras (2009)

Let *E*, *F*, *G*, and *H* be the midpoints of the consecutive sides of a quadrilateral *ABCD*. What type of quadrilateral is *EFGH*? (Quadrilateral *EFGH* is known as the medial quadrilateral of *ABCD*.)

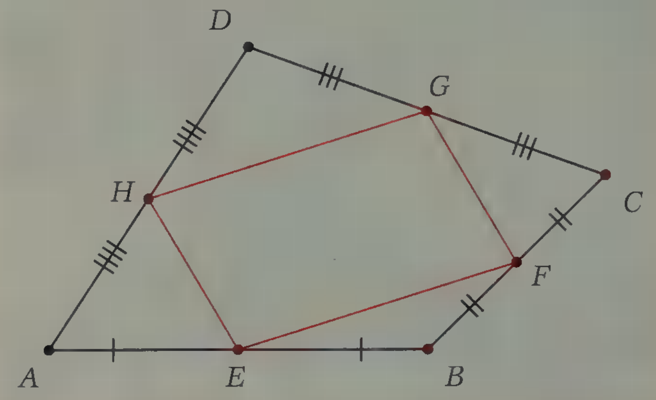


Fig. 1 The Varignon problem investigates the Varignon theorem.

validated their final conjectures with proofs.

The first proof developed was the proof of the Varignon theorem itself:

Theorem: The midpoints *E*, *F*, *G*, and *H* of the consecutive sides of a quadrilateral *ABCD* are the vertices of a parallelogram.

Proof: Construct diagonals *AC* and *BD* (see **fig. 2**). *FG* is a midsegment of $\triangle BCD$ because *F* and *G* are midpoints of two sides of $\triangle BCD$. By the midsegment-of-a-triangle theorem, $\overline{FG} \parallel \overline{BD}$. In like manner, $\overline{BD} \parallel \overline{EH}$. By the transitive property of parallelism, $\overline{FG} \parallel \overline{EH}$. By the same reasoning, $\overline{EF} \parallel \overline{HG}$. Thus, *EFGH* is a quadrilateral whose opposite sides are parallel—that is, a parallelogram.

This proof is remarkably simple, beautiful, and economical.

POSE AND SOLVE CONVERSE PROBLEMS

The problem-posing framework that the class used includes formulating a converse problem as a systematic strategy (Contreras 2003, 2009). Thus, it was natural to propose and investigate Varignon converse problems. To start, Daniel (all names are pseudonyms) proposed the following problem: If the inside (medial) quadrilateral is a parallelogram, what can you say about the original quadrilateral? The instructor thought that this problem could be reformulated in a self-contained form—that is, as a full formal problem statement that includes all the known and unknown attributes. After some discussion, the whole class accepted the following formulation of the problem:

Converse problem: *E*, *F*, *G*, and *H* are the midpoints of the consecutive sides of a quadrilateral *ABCD*. If *EFGH* is a parallelogram, what type of quadrilateral is *ABCD*?

Some students argued that the shape of *ABCD* depended on the shape of *EFGH*. Daniel, on the other hand, contended that *ABCD* would always be a quadrilateral, regardless of the shape of *EFGH*. In addition, he recognized that, under special conditions, *ABCD* could be any of the quadrilaterals listed in the first column of **table 2**. He commented that if *EFGH* were a rhombus, then *ABCD* had to be an isosceles trapezoid. Daniel's last statement was not correct, but I decided not to intervene at this moment.

After some debate, the class agreed that *ABCD* would always be a generic quadrilateral if *EFGH* were a generic parallelogram. This discussion opened the door for investigating the nature of *ABCD* for special cases of the medial parallelogram *EFGH*. Capitalizing on the information displayed in **table 2**, the students formulated the following converse problems:

(*a*) *E*, *F*, *G*, and *H* are the midpoints of the sides of a quadrilateral *ABCD*. If *EFGH* is a rhombus, what type of quadrilateral is *ABCD*?
(*b*) Let *ABCD* be a quadrilateral. If the medial quadrilateral of *ABCD* is a rectangle, investigate what type of quadrilateral *ABCD* is.
(*c*) If the medial quadrilateral of a quadrilateral *ABCD* is a square, what type of quadrilateral is *ABCD*?

An additional converse problem was formulated as a proof: If the medial quadrilateral *EFGH* of a quadrilateral *ABCD* is a square, prove that *ABCD* is a square. It is worthwhile to highlight this problem because students do not spontaneously generate proof problems (Contreras and Martínez-Cruz 1999). However, it is an ill-posed problem, as we will see, because *ABCD* is not necessarily a square.

Table 2 Quadrilaterals *ABCD* Generate Medial Quadrilaterals *EFGH*

Quadrilateral *ABCD*	Quadrilateral *EFGH*
Parallelogram	Parallelogram
Rectangle	Rhombus
Rhombus	Rectangle
Square	**Square**
Isosceles trapezoid	**Rhombus**
Kite	**Rectangle**
Trapezoid	Parallelogram
Quadrilateral	Parallelogram

Investigating the solutions to these seemingly simple converse problems appeared to be a trivial task at first. Using **table 2**, all students conjectured initially that the solutions to these problems were, respectively, an isosceles trapezoid, a kite, and a square. In other words, the students assumed that the theorems suggested by rows 4 through 6 in **table 2** were biconditional statements. Students chose a biconditional based on row 5, rather than on row 2, because they recognized the rectangle as a special case of the isosceles trapezoid. Similarly, a rhombus is a special case of a kite. The students' misconception arose because they thought that all types of special quadrilaterals, except the cyclic quadrilateral, were listed in the first column of **table 2**.

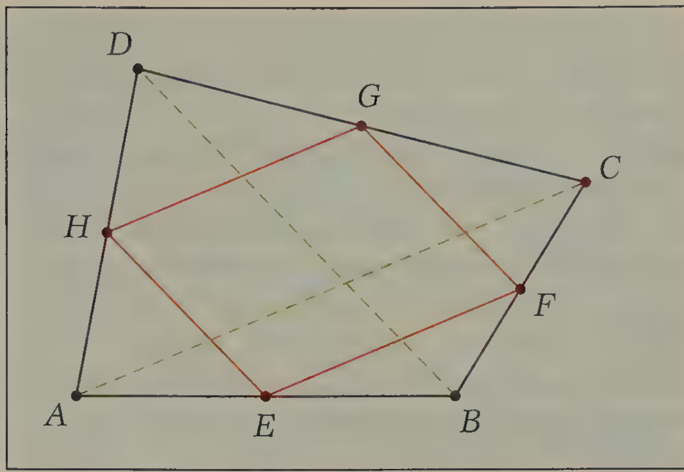
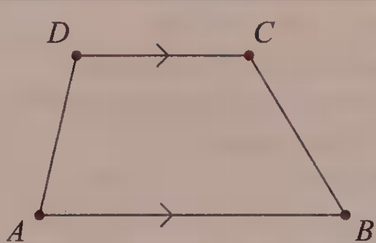
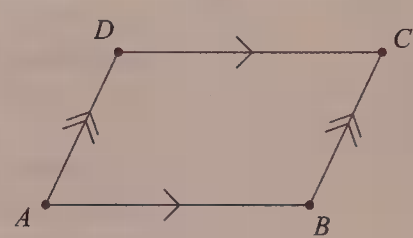
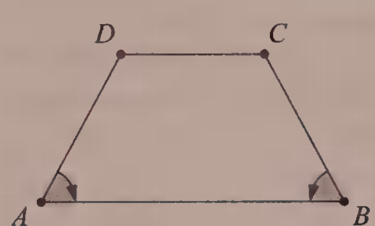
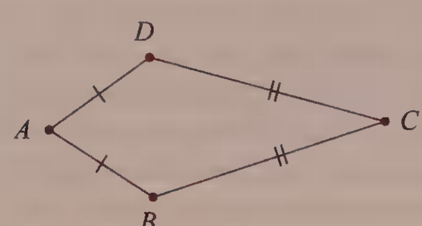
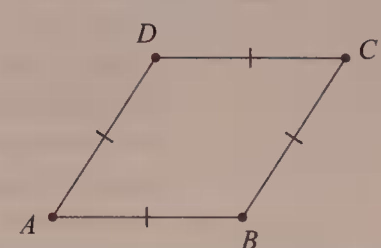


Fig. 2 A proof of the Varignon theorem involves the diagonals of the quadrilateral.

NEW STRATEGIES ARE NEEDED

The instructor challenged students to support their conjectures with a mathematical proof or with GSP, but they were unable to do so because their conjectures were not valid. All students stated their hypothesis and conclusion correctly, but they were unsure how to use GSP to investigate the converse problems. For example, to investigate converse problem (a), some students started with an isosceles trapezoid *ABCD* and then constructed its medial quadrilateral without realizing that they should have started with a rhombus *EFGH* and then constructed quadrilateral *ABCD*. When I asked them to justify their process, some said that they were certain that the quadrilateral *ABCD* with a rhombus as its medial quadrilateral would be an isosceles trapezoid, but they did not know how to prove this assertion

Table 3 Definitions and Examples of Controversial Special Quadrilaterals

Quadrilateral	Definition	Generic Example	Special Example
Trapezoid	A quadrilateral with at least one pair of parallel sides		
Isosceles trapezoid	A trapezoid with at least one pair of congruent base angles		
Kite	A quadrilateral with two nonoverlapping pairs of adjacent congruent sides		

or use GSP to support it. Other students said that they did not know how to use GSP to investigate a converse. Still others recognized that they "jumped" to the GSP construction without realizing that their configuration did not investigate the converse.

After this unsuccessful first attempt, Edna suggested starting with a generic quadrilateral *ABCD*, constructing its medial quadrilateral *EFGH*, and then dragging one or more vertices of *ABCD* until *EFGH* looked like the intended medial quadrilateral (e.g., a rhombus, rectangle, or square). The students then drew diagrams that supported their initial conjectures (see **fig. 3**). At this moment, I decided to intervene and challenged the class to try to draw or construct a counterexample that would refute their conjectures about *ABCD*. Most students, after some attempts, generated diagrams like those displayed in **figure 4**. The students were surprised to discover that their initial conjectures were false; their findings were in contradiction to their original expectation.

Characterizing each quadrilateral *ABCD* in **figure 4** puzzled students at first because the quadrilaterals did not seem to have distinctive properties. Students measured the sides and angles, but this approach did not reveal the defining property. The instructor asked students what else they could measure. After testing some additional unsuccessful guesses (e.g., areas and perimeters), Samantha exclaimed, "The diagonals!"

The dragging and measurement capabilities of GSP enabled students to formulate the following conjectures:

Conjecture (*a*): If the medial quadrilateral of a quadrilateral *ABCD* is a rhombus, then *ABCD* is a quadrilateral with congruent diagonals (i.e., an equidiagonal quadrilateral) (see **fig. 5a**).

Conjecture (*b*): If the medial quadrilateral *EFGH* of a quadrilateral *ABCD* is a rectangle, then *ABCD* is a quadrilateral with perpendicular diagonals (i.e., an orthodiagonal quadrilateral) (see **fig. 5b**).

Conjecture (*c*): The medial quadrilateral of a quadrilateral *ABCD* is a square if and only if *ABCD* is a quadrilateral with congruent and perpendicular diagonals (i.e., an equiorthodiagonal quadrilateral) (see **fig. 5c**).

Tammy formulated her conjecture in biconditional form (see conjecture [*c*]). Her approach is not surprising because the class had previously investigated properties of special quadrilaterals in biconditional form. Intrigued by their unforeseen discoveries, some students took on the challenge to construct proofs to promote these conjectures to theorems. Despite their persistence, however, they were not able to see the connection between the proof of the Varignon theorem and the proofs of the special converses. When some students asked for a hint, I suggested revisiting the proof of the Varignon theorem.

BACK TO THE BEGINNING

After some period of reflection, most students noticed the similarities between **figure 2** and **figure 5**. In **figure 5a**, for example, they recognized that FG was a midsegment of $\triangle BCD$ and concluded that $FG = BD/2$ by the midsegment-of-a-triangle theorem. After thinking further, some students also concluded that $GH = AC/2$ by the same reasoning. Then Nadine said, "Aha, I got it." The instructor asked her to explain her moment of enlightenment, and she replied: "Since $FG = GH$, we know that $BD/2 = AC/2$, which means $BD = AC$, so the diagonals of quadrilateral *ABCD* are congruent."

Other students crafted similar arguments to justify the second and third conjectures. For example, to prove conjecture (*b*), Martin reasoned along the following lines:

> *HG* and *EH* are midsegments of triangles *ACD* and *BDA*, respectively [see **fig. 5b**]. By the midsegment theorem, $\overline{HG} \parallel \overline{AC}$ and $\overline{EH} \parallel \overline{BD}$. So *HJIK* is a parallelogram. $\angle EHG$ is a right angle because *EFGH* is a rectangle. Hence, $\angle KIJ$ is also a right angle because the opposite angles of a parallelogram are congruent. In other words, the diagonals of quadrilateral *ABCD* are perpendicular.

To justify one component of conjecture (*c*), Nadine developed an elegant proof along the following lines:

> *EFGH* is a square [see **fig. 5c**]. We know that *ABCD* is a quadrilateral with perpendicular diagonals [proof of conjecture (*b*)] because a square is a rectangle. Because a square is a rhombus, the diagonals of *ABCD* are also congruent [proof of conjecture (*a*)]. So *ABCD* is an equiorthodiagonal quadrilateral because it has congruent and perpendicular diagonals.

At this juncture, the instructor asked the class whether we had proved completely the theorem suggested by conjecture (*c*) Tammy said that the conjecture was really a biconditional statement and that, as such, both statements need to be proved. Using the same reasoning used to prove Varignon's theorem, Tammy proved that the medial quadrilateral of an equiorthodiagonal quadrilateral was a square. The class then formulated biconditional statements for conjectures (*a*) and (*b*), whose proofs were assigned as homework. Needless to say, the students were surprised that both the Varignon theorem and its special converses were connected through a proof using the midsegment theorem.

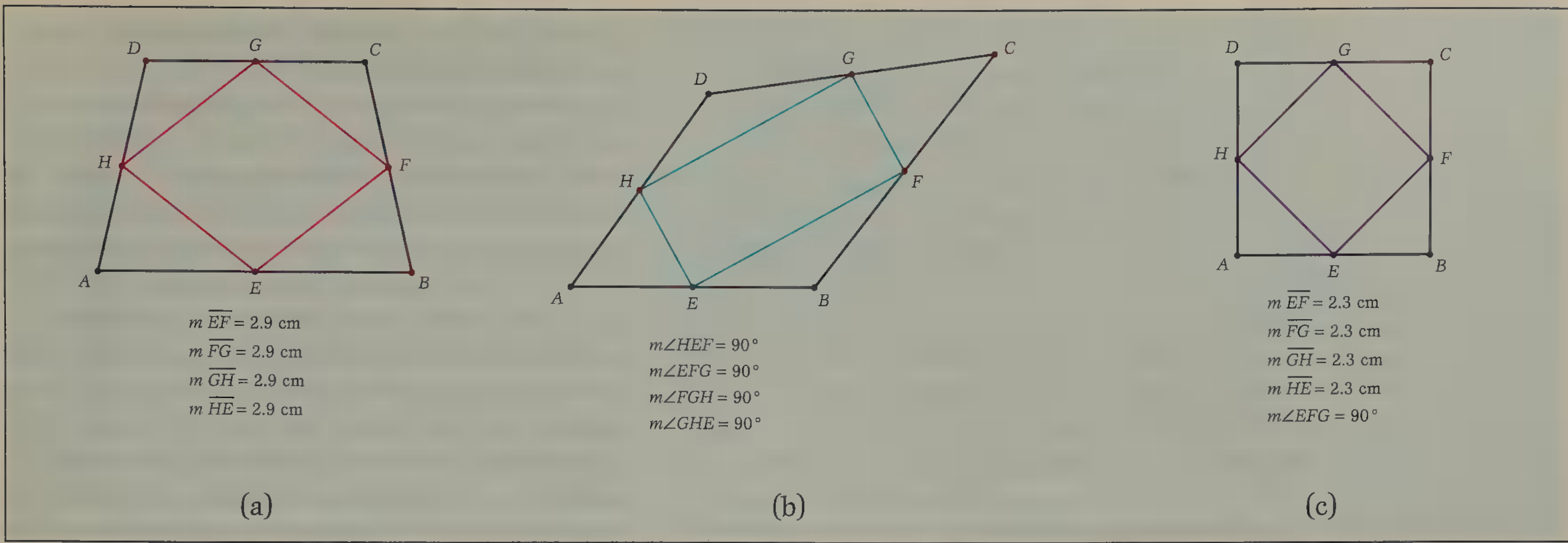


Fig. 3 Students create quadrilaterals to support converse statements suggested by **table 2**. The containing quadrilateral of a medial rhombus is an isosceles trapezoid (**a**); the containing quadrilateral of a medial rectangle is a kite (**b**); and the containing quadrilateral of a medial square is a square (**c**).

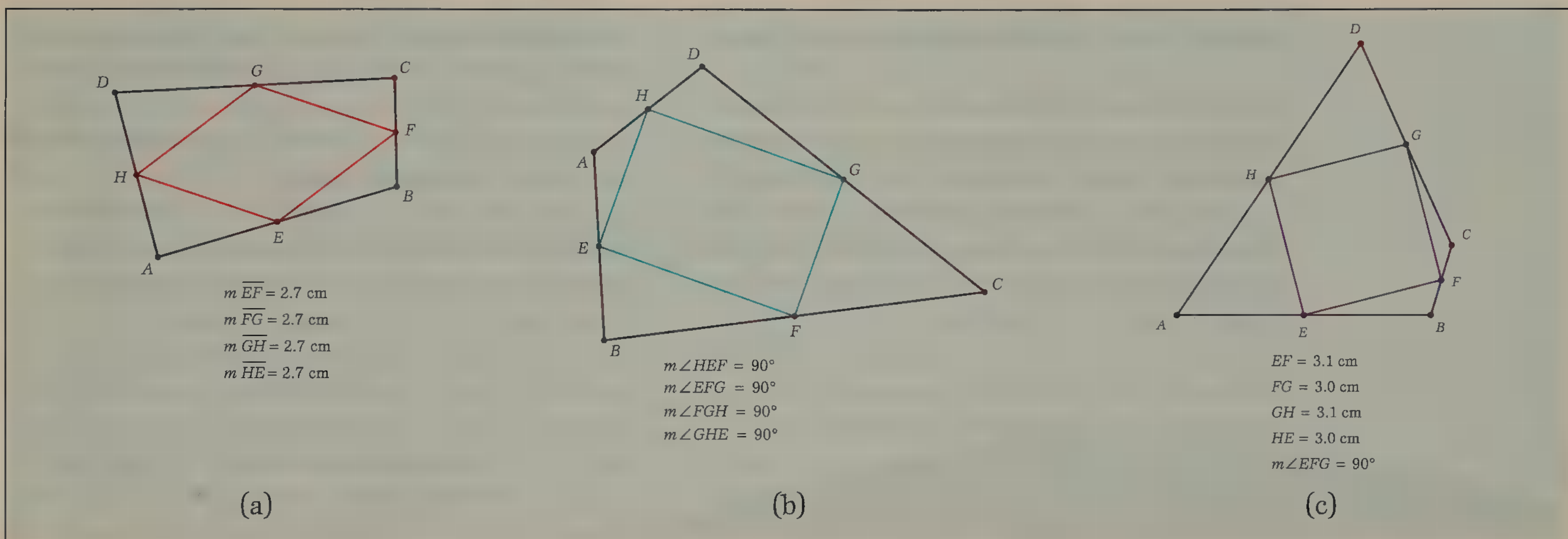


Fig. 4 Students create quadrilaterals to refute the converse statements suggested by **table 2**. The containing quadrilateral of a medial rhombus is not an isosceles trapezoid (**a**); the containing quadrilateral of a medial rectangle is not a kite (**b**); and the containing quadrilateral of a medial square is not a square (**c**).

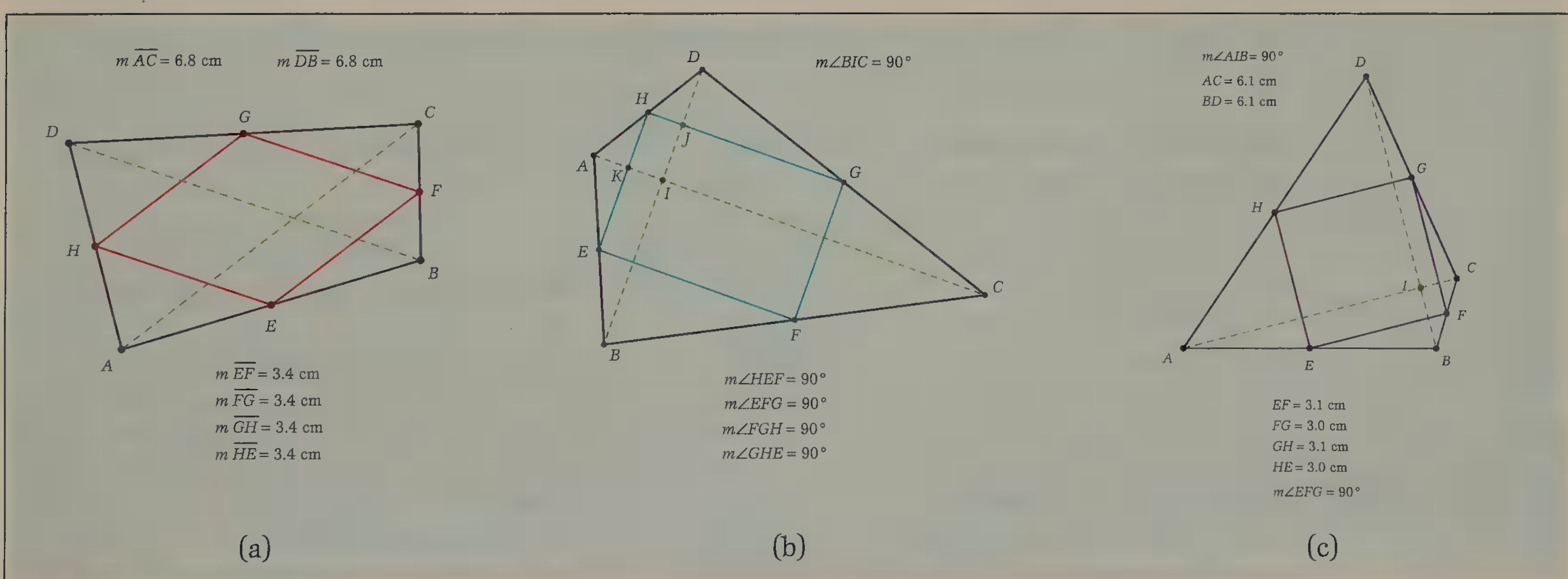


Fig. 5 Students formulate conjectures relating the shape of the medial quadrilateral to the diagonals of the containing quadrilateral. The quadrilateral for which *EFGH* is a rhombus has congruent diagonals (**a**); the quadrilateral for which *EFGH* is a rectangle has perpendicular diagonals (**b**); and the quadrilateral for which *EFGH* is a square has congruent and perpendicular diagonals (**c**).

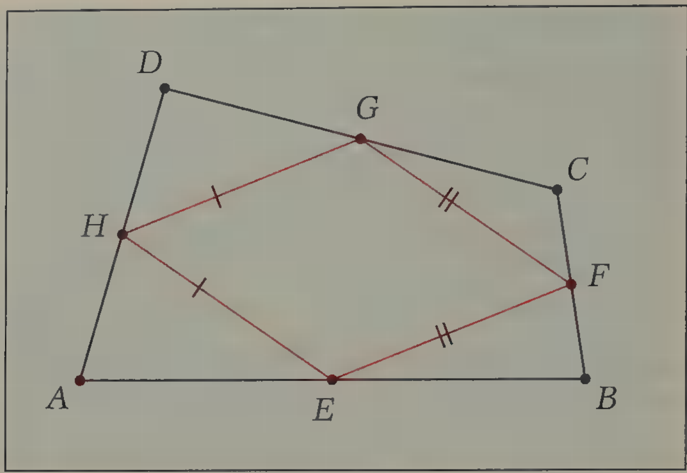


Fig. 6 Can the kite *EFGH* be other than a rhombus?

After this discussion, the instructor asked the class whether there exists a quadrilateral that has as its medial quadrilateral a (generic) kite with $HE \neq EF$ (see **fig. 6**). After some deliberation, a couple of students stated that the kite had to be a rhombus. They offered the following justification:

> We know that *EFGH* has to be a parallelogram [because the medial quadrilateral of any quadrilateral is a parallelogram]. We know that $\overline{HE} \cong \overline{GF}$ and that $\overline{HG} \cong \overline{EF}$ because the opposite sides of a parallelogram are congruent. So $\overline{GF} \cong \overline{HE} \cong \overline{HG} \cong \overline{EF}$, and *EFGH* is a quadrilateral with all sides congruent; thus, it is a rhombus.

Asked by the instructor to pose additional problems, Albert asked whether a converse problem could be formulated by asking what type of points *E*, *F*, *G*, and *H* were. That is, if *EFGH* is a parallelogram, are *E*, *F*, *G*, and *H* always midpoints of the respective sides of quadrilateral *ABCD*? This question led the class into another path of inquiry. The members of the class continued their problem-posing and -solving adventures into uncharted territory, but their conquests and defeats shall be told another time.

ENCOURAGE DISCUSSION AND ACCOUNTABILITY

The investigation described in this article spanned one and a half periods (about ninety-five minutes total). The students worked in pairs to formulate the problems, conjectures, and proofs, but each student had his or her own computer. To keep a record, the instructor wrote or sketched the students' problems, conjectures, and proofs on the board and asked the whole class to provide comments, criticisms, and suggestions to improve the content and language. Those discussed here are the final product of collaboration among the members of the class, with minimum intervention from the instructor. To encourage individual learning accountability, students were asked to pose and solve a Varignon converse problem in an examination. All students' responses were judged to be correct.

As noted, these students were proficient in the use of GSP, and all learners should be facile with such software before performing these types of investigations. Learning how to operate the necessary commands (constructing, measuring, dragging, hiding, etc.) can be overwhelming and can distract students from focusing on the investigations at hand. Also, to foster inquiry and collaboration, teachers should have in place the following norms adapted from *Professional Standards for Teaching Mathematics* (NCTM 1991): Encourage and allow sufficient time for students to think, reflect, and grapple with their mathematical ideas individually, in small groups, and during whole-class discussions; consistently ask students to clarify and justify their solutions to problems, whether reasonable or not, to one another and the whole class; value and respect students' ideas so they are not intimidated to express their thinking; and, finally, monitor how students, individually or in small groups, are proceeding in their solution and offer hints or resources when appropriate.

Varignon converse problems are a suitable topic for both high school students and preservice secondary school mathematics teachers. Further, their simplicity and richness engage learners in a diversity of mathematical practices recommended by the Common Core's Standards for Mathematical Practice (CCSSI 2010). The knowledge needed to solve them, the midsegment-of-a-triangle theorem, is a standard curriculum topic of high school geometry. Also, this theorem is explicitly mentioned in the CCSSM (2010, p. 76). At the same time, the Varignon problem offers opportunities to engage students in a plethora of CCSS mathematical practices: persevere in solving problems, construct viable arguments and critique the reasoning of others, use appropriate tools, and make use of appropriate reasoning. Teachers may want to select the problem situations and the approach that are more appropriate for their students' needs and backgrounds.

PROMOTE GOOD MATHEMATICAL HABITS

When asked what they had learned as a result of this activity, the students provided a variety of responses. One stated, "Every time we are faced with a mathematical problem or theorem, we should try to ask questions about converses and try to prove them." Another said, "Sometimes converses could be challenging, or trivial, but [it] is always worth the trouble trying to solve them." Still another added, "While GSP may not give you all the answers, it gives you sometimes directions or ideas of what to do next. It can help to construct a counterexample."

Promoting mathematical habits of mind should be a goal of teaching and learning mathematics (Cuoco, Goldenberg, and Mark 1996; Goldenberg 1996; Goldenberg, Shteingold, and Feurzeig 2003; Mark et al. 2010; Levasseur and Cuoco 2003). Two mathematical habits of mind that may empower students to pose and solve problems are formulating converse problems and proving.

By engaging systematically in problem posing—that is, posing a converse problem and other problems for every given problem, if possible—all of us—students, teachers, and mathematics teacher educators—may develop a genuine disposition toward doing or reinventing mathematics. In doing so, we will experience the delight of posing and solving new problems, even if they are new only to us.

ACKNOWLEDGMENT

I wish to thank Armando Martínez-Cruz for his comments, criticisms, and suggestions on earlier drafts of this article.

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Contreras, José N. 2003. "A Problem-Posing Approach to Specializing, Generalizing, and Extending Problems with Interactive Geometry Software." *Mathematics Teacher* 96 (4): 270–76.

———. 2009. "Generating Problems, Conjectures, and Theorems with Interactive Geometry: An Environment for Fostering Mathematical Thinking by All Students." In *Mathematics for Every Student: Responding to Diversity, Grades 9–12*, edited by Alfinio Flores and Carol E. Malloy, pp. 71–83. Reston, VA: National Council of Teachers of Mathematics.

Contreras, José N., and Armando M. Martínez-Cruz. 1999. "Examining What Prospective Secondary Teachers Bring to Teacher Education: A Preliminary Analysis of Their Initial Problem-Posing Abilities within Geometric Tasks." In *Proceedings of the Twenty-First Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, vol. 2, edited by Fernando Hitt and Manuel Santos, pp. 413–20. Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Cuoco, Al, E. Paul Goldenberg, and June Mark. 1996. "Habits of Mind: An Organizing Principle for

Mathematics Curriculum." *Journal of Mathematical Behavior* 15 (4): 375–402.

Goldenberg, E. Paul. 1996. "'Habits of Mind' as an Organizer for the Curriculum." *Journal of Education* 178 (1):13–34.

Goldenberg, E. Paul, Nina Shteingold, and Nannette Feurzeig. 2003. "Mathematical Habits of Mind for Young Children." In *Teaching Mathematics through Problem Solving: Prekindergarten–Grade 6*, edited by Frank K. Lester Jr. and Randall I. Charles, pp. 15–29. Reston, VA: National Council of Teachers of Mathematics.

Levasseur, Kenneth, and Al Cuoco. 2003. "Mathematical Habits of Mind." In *Teaching Mathematics through Problem Solving: Grades 6–12*, edited by Harold L. Schoen and Randall I. Charles, pp. 27–37. Reston, VA: National Council of Teachers of Mathematics.

Mark, June, Al Cuoco, E. Paul Goldenberg, and Sarah Sword. 2010. "Contemporary Curriculum Issue: Developing Mathematical Habits of Mind." *Mathematics Teaching in the Middle School* 15 (9): 505–9.

Movshovits-Hadar, Nitsa. 1988. "School Mathematics Theorems—An Endless Source of Surprise." *For the Learning of Mathematics* 8 (3): 34–40.

National Council of Teachers of Mathematics (NCTM). 1991. *Professional Standards for Teaching Mathematics*. Reston, VA: NCTM.

Oliver, Peter. 2001a. "Pierre Varignon and the Parallelogram Theorem." *Mathematics Teacher* 94 (4): 316–19.

———. 2001b. "Consequences of the Varignon Parallelogram Theorem." *Mathematics Teacher* 94 (5): 406–408.

Pólya, George. 1945. *How to Solve It: A New Aspect of Mathematical Method*. Princeton, NJ: Princeton University Press.

JOSÉ N. CONTRERAS, jncontrerasf@bsu.edu, teaches mathematics and mathematics education courses at the undergraduate and graduate levels at Ball State University in Muncie, Indiana. He is interested in integrating problem posing and solving, technology, realistic contexts, modeling, history, proof, and aesthetic aspects of mathematics in teaching and teacher education.


FOCUS
COHERENCE
RIGOR

IMPLEMENTING THE COMMON CORE

APPLYING SHIFTS TO INSTRUCTION

A classroom vignette illustrates the design principles of focus, coherence, and rigor that frame the Common Core State Standards for Mathematics.

Angeline K. Gaddy, Shannon E. Harmon, Angela T. Barlow, Charles D. Milligan, and Rongjin Huang

I know that for the Common Core I need to rethink how I teach mathematics. But how do I get started? What should I concentrate on? What do I need to do differently?

As we have worked with teachers in professional development, these questions and others have led us to consider focus, coherence, and rigor as key areas in which teachers need to adjust their instruction. Making sense of these design principles has served as a springboard for the teachers in implementing the Common Core State Standards for Mathematics (CCSSM). Recognizing that others may be asking similar questions, we aim in this article to support teachers in not only understanding these design principles but also applying them as shifts to instruction. We begin by situating these principles within the context of NCTM's earlier Standards and CCSSM.

FOUNDATION FOR THE CCSSM

With such publications as *Curriculum and Evaluation Standards* (1989) and *Principles and Standards for School Mathematics* (2000), NCTM has played a significant role in defining a vision for school mathematics. In particular, the Curriculum Principle (NCTM 2000, pp. 14–16) described the need for students to learn important mathematics (focus) that is interconnected within and across grade levels (coherence). In addition, the Learning Principle (NCTM 2000, pp. 20–21) described the need for students not only to develop conceptual understanding and procedural fluency but also to apply this knowledge (rigor).

Building on this foundation, the creators of CCSSM used focus, coherence, and rigor as design principles to frame these standards (CCSSI 2010). That is, they sought to create a curriculum that included important interconnected mathematics and represented a balance of conceptual understanding,

procedural skill, and application. Understanding the need to support teachers in implementing CCSSM, educators and researchers have presented these design principles as *shifts*, or "key changes required by the Standards" (Student Achievement Partners 2012b, paragraph 2). Because many of us are unsure of what these standards entail in regard to our instruction (Gewertz 2013), these shifts—focus, coherence, and rigor—provide a means for understanding the instructional changes needed for implementing the Common Core.

To understand the shifts, we first offer a description of the terms *focus*, *coherence*, and *rigor*. Next, we look at how to apply these shifts in a classroom setting. Finally, we solidify this understanding through a vignette that illustrates this application of the shifts.

Focus

To meet the expectations of CCSSM, teachers must *focus* on the mathematical ideas embedded within the standards. Attention should not be limited to the development of procedural skill. Rather, conceptual understanding along with application of mathematical ideas should play a key role in students' learning.

Coherence

CCSSM emphasizes the mathematics progression across grade levels as well as the links among mathematics topics within conceptual categories. The term *coherence* is used to describe these attributes of CCSSM. This means that "each standard is not a new event, but an extension of previous learning" (Student Achievement Partners 2012a, paragraph 2). Therefore, we must use the learning trajectories present within CCSSM to inform our teaching.

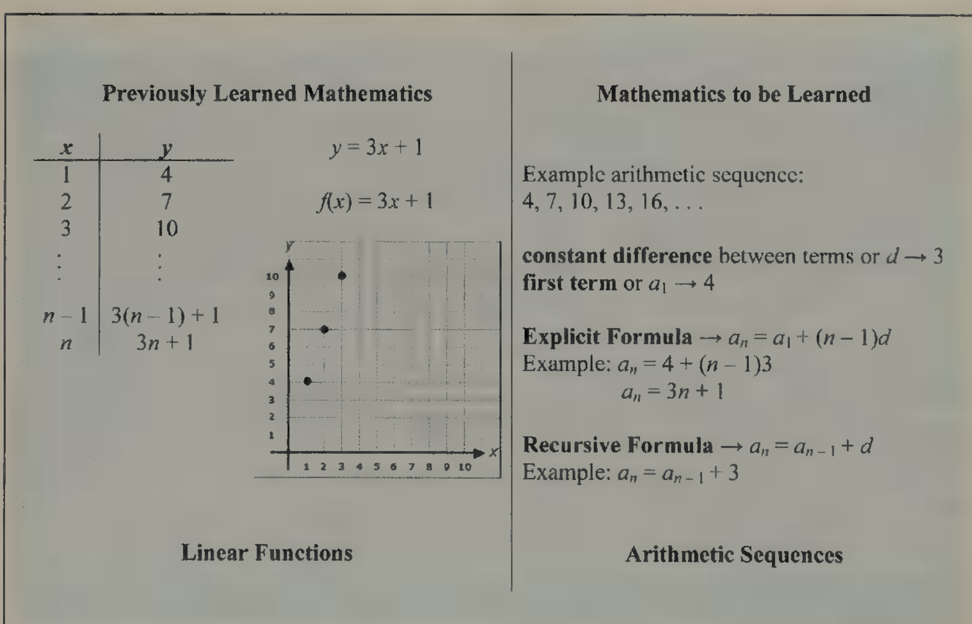


Fig. 1 The mathematics of linear functions prepares students to learn about arithmetic sequences.

Rigor

CCSSM calls for mathematics to be taught with a level of *rigor* that includes equitable attention to conceptual understanding, procedural fluency, and application. As teachers, we must purposefully plan learning opportunities for students with this in mind. Purposeful planning includes attention to the task and its implementation. Tasks with high cognitive demand (Stein et al. 2000) hold the potential for students to achieve rigor through the development of conceptual understanding and application of mathematical ideas.

The implementation process, however, is vital. The key idea here is that when teachers focus on task selection and implementation, students will move forward in meeting rigor. A classroom example representing these ideas follows.

APPLYING SHIFTS TO INSTRUCTION

The first step toward understanding any standard is to focus on its major mathematical ideas. Therefore, we begin by identifying the conceptual understandings, procedural skills, and mathematical applications embedded within a given standard. To demonstrate this idea, we will use CCSSM Standard F-BF.2 (Functions, Building Functions), which states that students will "write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms" (CCSSI 2010, p. 70). Students will understand that sequences can be represented in multiple ways, including explicit and recursive formulas. Students will gain skill in writing both explicit and recursive formulas for given sequences. In addition, students will apply these understandings and skills to model real-world situations in which sequences occur.

Recognizing the coherence represented in CCSSM, our next step is to consider the progression of the mathematics for this standard across grade levels. According to the standards, students' learning experiences in mathematics before high school should have addressed characteristics of a function; different representations of functions; differences between linear and nonlinear functions; and functions as a tool for modeling linear relationships. A natural extension of linear functions is arithmetic sequences (see **fig. 1**). To support coherence, instruction should begin with these ideas and then move toward the development of geometric sequences.

Our final step is to consider the level of rigor that is necessary to meet the expectations of our sample standard. First, we must select a task with high cognitive demand whose mathematical goals align with the standard. For Standard F-BF.2, we selected the Pet Ward Construction problem

A national pet-hotel chain is planning to build units for a series of franchises. Each unit for small pets is a row of two-meter-by-two-meter square wards. The wards are connected as shown in the partial floor plan below:

Walls for these units come only in two-meter panels, and the number of two-meter panels needed depends on the number of wards to be included in the unit. Because the management plans to build many units of different sizes, the manager wants to have a rule relating the number of words and the number of panels.

Source: Heid et al. (1995), p. 50

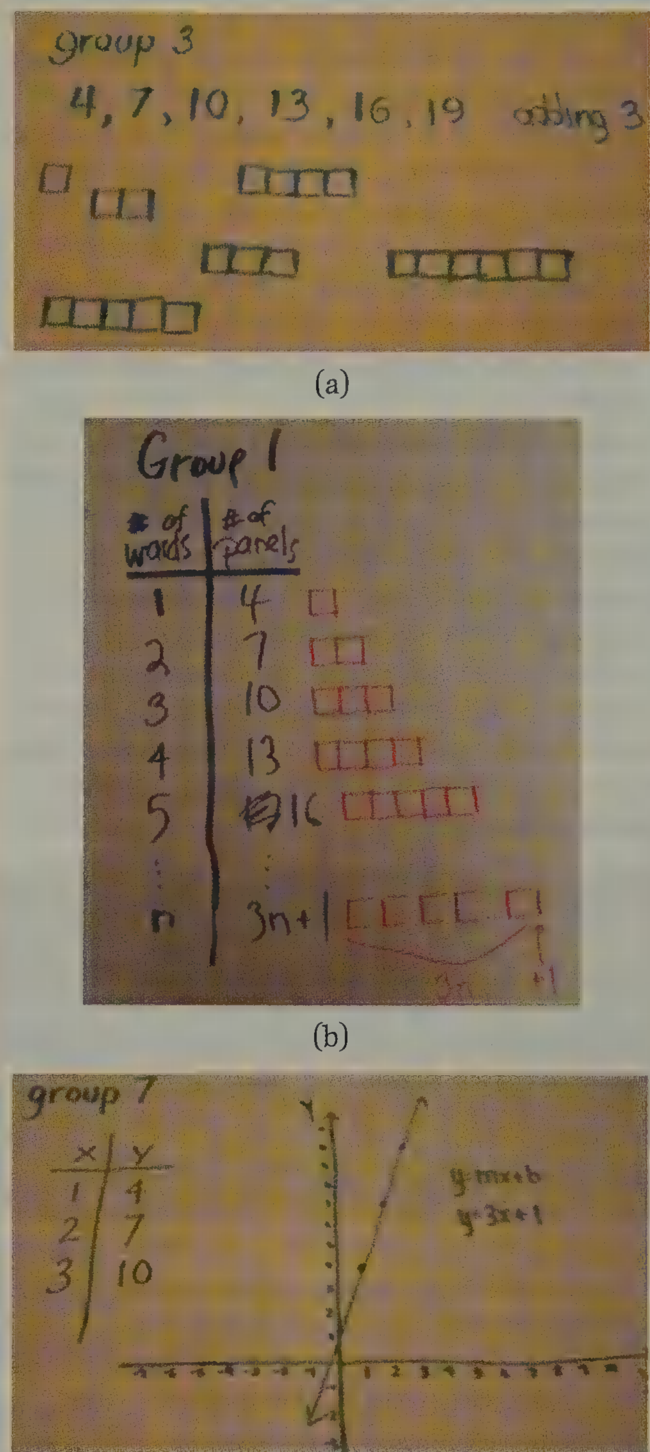
Fig. 2 The Pet Ward Construction problem is a task with high cognitive demand.

(Heid et al. 1995) (see **fig. 2**). This task supports the conceptual understanding to be developed regarding arithmetic sequences; thus, it is necessary to consider the implementation of the task. The following classroom vignette provides a view of the shifts in instruction.

A CLASSROOM VIGNETTE

After the students read the Pet Ward Construction problem, the teacher leads a discussion. As students share their observations about the image and problem, including how many panels are used and the possibility of other arrangements of the wards, the teacher makes comments such as "Nice observation," "Will this always be true?" and "Is there anything else we know?" After several students have contributed, the teacher asks a student to summarize what the problem is asking and then tells the class to work in groups to formulate a plan for solving the problem. As students work, the teacher distributes chart paper and markers for groups to record their ideas. She circulates around the room, attending to the strategies being used. Group 3's representation (see **fig. 3a**) is unique. Most groups produce work similar to that featured in **figure 3b**, although not all groups develop a generalized pattern. Two groups' work is similar to that shown in **figure 3c**.

After the groups have displayed their ideas on chart paper, the teacher brings the class together to discuss the different strategies. The strategies used by groups 1, 3, and 7 are representative of the strategies of the class as a whole, so the teacher selects the work of these three groups to present.



(a)

(b)

(c)

Fig. 3 Different representations show the pattern of adding 3.

***Teacher*:** Wow! I see several different strategies that we should think about. I will ask three groups to present their work. Then I will ask you all to think about how the strategies are alike and different. Let's start with group 3. Dave, will you present your group's strategy for solving this problem?

***Dave*:** We listed the number of panels for 1 ward and 2 wards and 3 wards and so on, and we noticed that we were adding 3, and we drew pictures.

***Teacher*:** Thank you, Dave. Let's hear from group 1. Anna?

***Anna*:** We drew a T-chart and put the number of wards on one side and the number of panels on the other side. We also drew pictures to help us

count the number of panels. We saw what Dave was talking about that the number of panels was going up by 3, but we wanted to be able to tell the number of panels based on the number of wards.

Teacher: Would someone from Anna's group tell us about this expression, $3n + 1$? Carra?

Carra: We looked at the wards as a bunch of Cs. So, see, in the picture, the shape of a ward is a C, and each has 3 panels. So you put together all of the Cs, and then you have to add 1 more panel at the end.

Teacher: OK. The last group I would like for us to hear from is group 7. Juan, will you present your group's work?

Juan: We also drew a T-chart but then made a graph so that we could find the slope and y-intercept. So we wrote an equation.

After the different strategies have been shared, the teacher leads a discussion of these. A sampling of questions that the teacher used to guide the discussion follows:

1. How are all three solution processes the same?
2. How are the solution processes different?
3. Why is it that we see the phrase "adding 3" on group 3's poster, but on the other two posters it looks like we are multiplying by 3?
4. If we needed to know how many panels to purchase for 100 wards, how would you figure that out using group 7's method? Using group 1's method? Using group 3's method?

As the end of class nears, the teacher asks the students to write their thoughts regarding the different solution methods. In thinking about the next day's lesson, she plans to build on the students' presentations by introducing the terms *arithmetic sequences* (as depicted in group 3's poster), *recursive formulas* (building on group 3's phrase of "adding 3"), and *explicit formulas* (building on group 1's generalized pattern and group 7's equation).

DISCUSSION OF VIGNETTE

This vignette demonstrates how the teacher purposefully pursued rigor by appropriately implementing a task and orchestrating the discussion of students' work (Stein et al. 2008). Initially, the teacher used questions to support students in making sense of the problem. Then she moved around the room, monitoring students' solution processes with a focus on selecting and sequencing different strategies for presentation. Finally, she asked students to explain and justify their solutions.

In sequencing the different strategies for presentation, the teacher chose to move from the least sophisticated strategy to the most sophisticated strategy. Group 3's strategy (see **fig. 3a**) was an "informal" but natural way of presenting the "adding 3" structure based on the students' previous experiences in pattern seeking. Featuring this strategy for the initial presentation not only validated the strategy but also likely supported any struggling students in developing a fundamental understanding of the mathematics represented in the problem. Next, the teacher chose to present group 1's work because its strategy represented the thinking of the majority of the class. Group 1's strategy (see **fig. 3b**) focused on expressing the "adding 3" pattern using an algebraic expression with the support of a geometric representation. Finally, the teacher presented group 7's strategy because it represented the most sophisticated strategy. This strategy involved using a linear equation and graph to represent the "adding 3" structure (see **fig. 3c**). Although this particular teacher chose to move from least to most sophisticated strategy, this sequence is not always necessary. For a discussion of purposeful sequencing of students' presentations, see Stein and Smith (2011).

THIS PROBLEM-BASED APPROACH TO INTRODUCING THE CONCEPT OF ARITHMETIC SEQUENCES ENHANCED STUDENTS' ABILITY TO APPLY THE KNOWLEDGE TO REAL-WORLD PROBLEMS.

Following the presentations, the discussions of these strategies supported students' in recognizing the repeated structure using different representations or tools. Moreover, questions 1 and 2 helped students understand that there are multiple ways to represent the same pattern (repeated structure) from different perspectives—focusing on the relationship between the current term and the previous term or focusing on the relationship between the current term and the term number. Question 3 led students to translate between the recursive formula and the explicit formula and determine the advantages and disadvantages of each. Finally, question 4 encouraged students to examine the power of using the explicit formula.

These discussions focused on the development of students' conceptual understanding of recursive and explicit formulas, their usefulness in modeling situations, and the translation between the two forms. Through this process, multiple ways of representing the same pattern of "adding 3"—tables, algebraic expressions, geometric representations, and graphs—supported students in developing procedural fluency with algebra and functions. This problem-based approach to introducing the concept of arithmetic sequences enhanced students' ability to apply the knowledge to real-world problems. The teacher demonstrated her intent to focus on the core concept of arithmetic sequences coherently. Exploring the task in this lesson built on students' previous knowledge of pattern seeking, algebraic expressions, and linear equations and graphs and developed students' understanding of recursive and explicit formulas for arithmetic sequences. In this way, the lesson demonstrates coherence. This exploration laid the foundation for learning arithmetic sequences formally and extensively in the next lessons. In this way, the three Common Core shifts—focus, coherence, and rigor—are demonstrated.

SUPPORTING MATHEMATICS INSTRUCTION

For teachers to better understand instructional changes needed to meet CCSSM expectations, viewing instruction through the lenses of focus, coherence, and rigor is beneficial. By representing these shifts through descriptions and application, we aimed to support teachers in implementing CCSSM. Although these standards served as the impetus to the discussion, these shifts represent ideas that will support quality high school mathematics instruction regardless of the curriculum.

REFERENCES

Common Core State Standards Initiative (CCSSI). 2010. Common Core State Standards for Mathematics. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf

Gewertz, Catherine. 2013. "Teachers Say They Are Unprepared for Common Core." http://www.edweek.org/ew/articles/2013/02/27/22common_ep.h32.html?tkn=OSSFrEuPIlXnXkiT5p3vIlaFR0%2BG%2BZPg%2F4jk&cmp=ENL-EU-NEWS1

Heid, M. Kathleen, Jonathan Choate, Charlene Sheets, and Rose Mary Zbiek. 1995. *Algebra in a Technological World. Curriculum and Evaluation Standards for School Mathematics.* Addenda Series, Grades 9–12. Reston, VA: National Council of Teachers of Mathematics.

National Council of Teachers of Mathematics (NCTM). 1989. *Curriculum and Evaluation Standards for School Mathematics.* Reston, VA: NCTM.

———. 2000. *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Stein, Mary Kay, and Margaret S. Smith. 2011. *5 Practices for Orchestrating Productive Mathematics Discussions.* Reston, VA: National Council of Teachers of Mathematics.

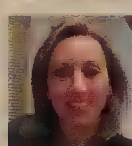
Stein, Mary Kay, Randi A. Engle, Margaret Schwan Smith, and Elizabeth K. Hughes. 2008. "Orchestrating Productive Mathematical Discussions: Five Practices for Helping Teachers Move beyond Show and Tell." *Mathematical Thinking and Learning* 10 (4): 313–40. doi:http://dx.doi.org/10.1080/10986060802229675

Stein, Mary Kay, Margaret Schwan Smith, Marjorie A. Henningsen, and Edward A. Silver. 2000. *Implementing Standards-based Mathematics Instruction: A Casebook for Professional Development.* New York: Teachers College Press.

Student Achievement Partners. 2012a. *Common Core Shifts for Mathematics.* http://www.achievethecore.org/downloads/E0702_Description_of_the_Common_Core_Shifts.pdf

———. 2012b. *Common Core: Math.* http://www.achievethecore.org/math-common-core/math-shifts/





ANGELINE K. GADDY, akg3c@mtmail.mtsu.edu, is a doctoral student studying mathematics education at Middle Tennessee State University in Murfreesboro. Formerly a secondary school mathematics teacher, she is interested in supporting preservice and in-service mathematics teachers in improving their teaching skills. **SHANNON E. HARMON,** shannon.harmon@mtsu.edu, works with both preservice and in-service teachers at MTSU. She is interested in providing teachers and teacher candidates with a vision of reformed-based teaching. **ANGELA T. BARLOW,** angela.barlow@mtsu.edu, is the director of the Mathematics and Science Education Ph.D. program at MTSU. She is interested in developing teachers' beliefs that align with reform-oriented instruction. **CHARLES D. MILLIGAN,** charles.milligan@mtsu.edu, is an assistant professor in the Educational Leadership Department at MTSU. He is a former principal and superintendent in K–12 education, and his research interests include evidence-based accountability. **RONGJIN HUANG,** rhuang@mtsu.edu, is an associate professor of mathematics education at MTSU. His research interests include mathematics classroom research, mathematics teacher education, and comparative mathematics education.

Unit Circles and Inverse Trigonometric Functions

A method to determine all the inverse trigonometric functions directly from the unit circle.

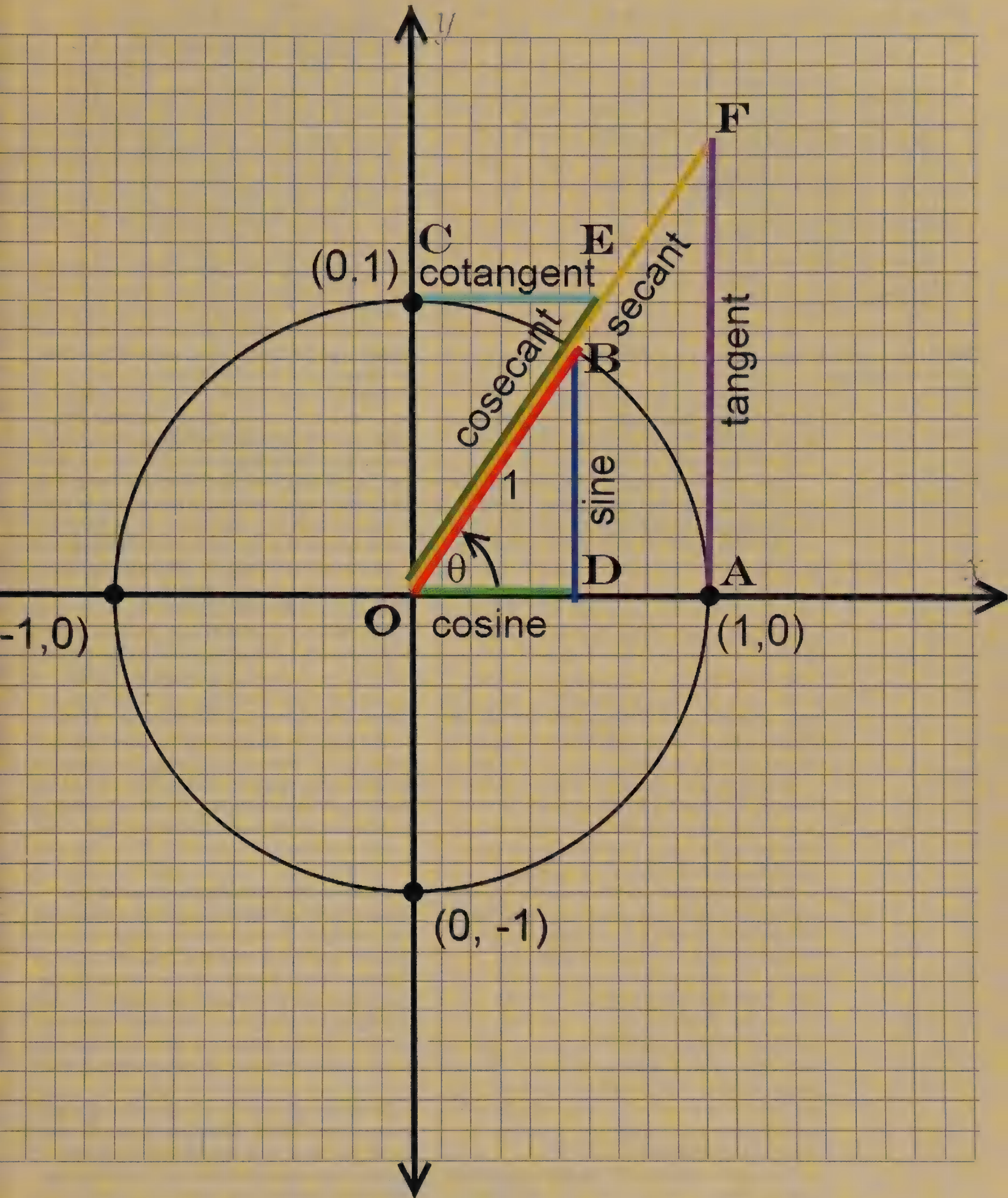
Azael Barrera

In most textbooks that are used to teach trigonometry with the unit circle, students learn to determine the trigonometric functions from the coordinates of the point (x, y), where the ray of a given angle intersects the circle and the values of the cosine and sine are the respective x- and y-coordinates. They also describe how to find the other functions from ratios and the reciprocal of the sine and cosine (Aufmann, Barker, and Nation 2008; Barnett, Ziegler, and Byleen 2011; Cohen, Lee, and Sklar 2005; Larson 2011; Stewart, Redlin, and Watson 2012; Swokowski and Cole 2009; Sullivan 2012; Trigsted 2012; Young 2010; Zill and Dewar 2012). One textbook (Stewart, Redlin, and Watson 2012) refers to the construction of the trigonometric functions from measured lengths formed by the angle's ray intersections but only as a historical note. None of the textbooks reviewed described the direct use of the unit circle to find all inverse trigonometric functions.

Historical accounts of trigonometry refer to the works of many Indian and Arab astronomers on the origin of the trigonometric functions as we know them now, in particular Abu al-Wafa (ca. 980 CE), who determined and named all known trigonometric functions from segments constructed on a regular circle and later on a unit circle (Moussa 2011; Bressoud 2010). Determinations of the trigonometric function values from right-triangle ratios came later, with Johann Muller Regiomontanus's work *De Triangulis Omnimodis* of 1533 (Van Brummelen 2009) and others. Recently scholars have advocated teaching trigonometry using the unit circle within the Greek and the Arabic-Indian historical context (Bressoud 2010).

In my trigonometry class, my students learn to determine all trigonometric functions using only the unit circle. The unit circle construct that I use (see **opening art**) was adapted from Abu al-Wafa (Moussa 2011) and was later revived by Marolois (1627).

Students first draw the ray of a given angle on a unit circle in the standard position using a protractor; then they draw the segments that intersect



y
F
C
E
(0,1)
cotangent
secant
tangent
cosecant
B
1
sine
θ
D
A
O
cosine
-1,0)
(1,0)
(0, -1)

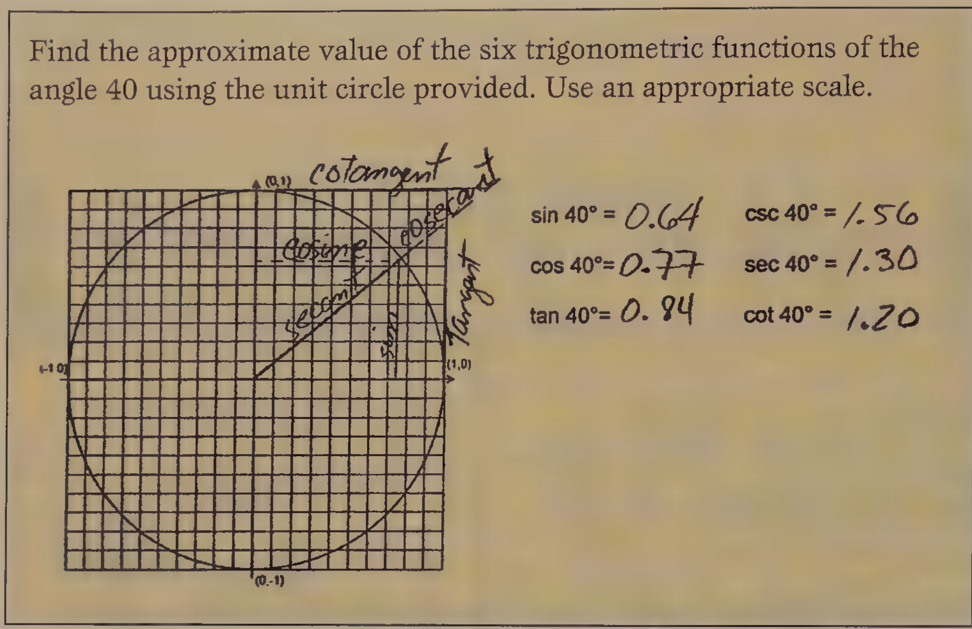


Fig. 1 A student's work shows measurement of lengths representing the six trigonometric functions.

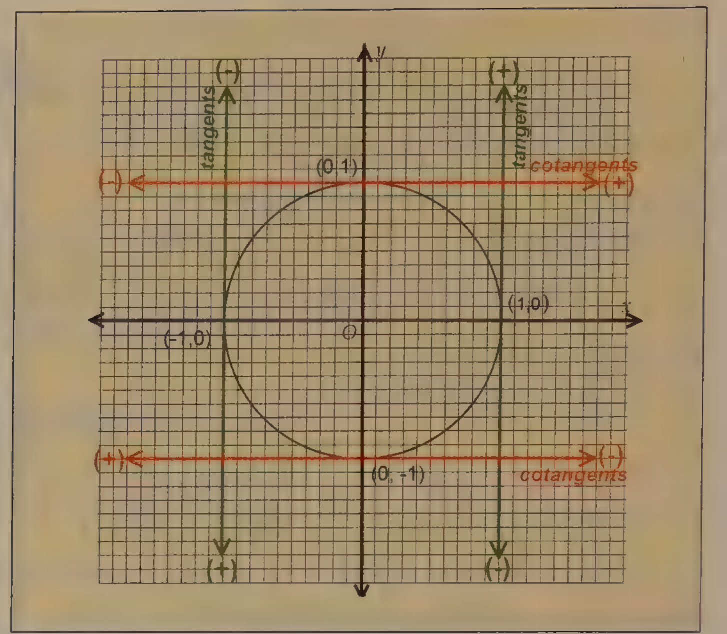


Fig. 2 The signs of function values are determined using this scheme.

the angle's ray. Finally, they measure the lengths of the six segments formed; **figure 1** shows an example of a student's work.

Signs of the functions in the different quadrants in the unit circle are determined using vertical tangent lines at (1, 0) and at (–1, 0), or $x = 1$ and $x = -1$, which we call *tangents*, and horizontal tangent lines at (0, 1) and at (0, –1), or $y = 1$ and $y = -1$, which we call *cotangents* (see **fig. 2**). The tangents line at (1, 0) corresponds to positive values upward and negative values downward, whereas the line at (–1, 0) corresponds to negative values upward and positive values downward. The cotangents line at (0, 1) corresponds to positive values to the right and negative values to the left, whereas the line at (0, –1) corresponds to positive values to the left and negative values to the right (see **fig. 2**).

After finding all trigonometric functions, the students are asked to check their results by using a calculator. To facilitate their measurements, my students use unit circle worksheets that are either unmarked or marked in degrees or radians.

Next, students learn to determine the values of the inverse trigonometric functions directly from the unit circle. To do so, I designed the following method based exclusively on the unit circle.

THE INVERSE UNIT CIRCLE METHOD

To find the value of the angles or arcs corresponding to inverse trigonometric functions, my students work with a method derived from the trigonometric lengths on the unit circle as described above. They use a special unit-circle worksheet to facilitate their measurements (this activity sheet is available at **www.nctm.org/mt058**).

The upper-half unit-circle worksheet includes only quadrants I and II and is used to find the inverse cosine with range $[0, \pi]$, the inverse cotangent with range $(0, \pi)$, or the inverse secant with range $[0, \pi/2) \cup (\pi/2, \pi]$. The right-half unit-circle worksheet includes only quadrants I and IV and is used to find the inverse sine with range $[-\pi/2, \pi/2]$, the inverse tangent with range $(-\pi/2, \pi/2)$, or the inverse cosecant with range $[-\pi/2, 0) \cup (0, \pi/2]$ and can be used for the alternate inverse cotangent with range $(-\pi/2, 0) \cup (0, \pi/2]$. A unit circle worksheet for inverse trigonometric functions marked in radians is also available to students.

The limited domain and range that make the original function and its respective inverse one-to-one functions are used to find the angle for each inverse function. I describe next how to use the unit circle to find each inverse function. The figures shown were adapted from exercises done in class with the students. The argument, or input, of the inverse function is a numerical value for which the students are asked to find the corresponding angle or arc that represents the output value of the inverse function.

The Inverse Sine, Cosine, and Tangent

The right-half worksheet is used to find the inverse sine (see **fig. 3a**). The given value of the argument of the inverse sine is located on the y-axis, and a horizontal line is drawn to intersect the circle. For the inverse cosine, the upper-half worksheet is used (see **fig. 3b**). The argument is located on the x-axis, and a vertical line is drawn to intersect the circle. In both cases, the angle's ray is traced from the origin to the corresponding intersection.

In the case of the inverse tangent, the right-half unit-circle worksheet is used (see **fig. 3c**). A vertical tangent line $x = 1$ is drawn. This is one of the tangents line mentioned before. A point is located on that tangents line, at a distance from (1, 0) equal to the given value of the argument of the inverse

tangent. A ray is drawn from the origin to that point. In all cases, the resulting angle is measured with a protractor from the x-axis in the standard way.

Examples of students' work are shown for inverse sine: $\sin^{-1}(0.8) = \arcsin(0.8) \approx 53°$ and $\sin^{-1}(-0.3) = \arcsin(-0.3) \approx -17°$ (see **fig. 3a**); for inverse cosine: $\cos^{-1}(0.8) = \arccos(0.8) \approx 37°$ and $\cos^{-1}(-0.6) = \arccos(-0.6) \approx 127°$ (see **fig. 3b**); and last for inverse tangent: $\tan^{-1}(0.5) = \arctan(0.5) \approx 26°$, and $\tan^{-1}(-1.5) = \arctan(-1.5) \approx -60°$ (see **fig. 3c**).

The Inverse Cotangent

There are two usual range definitions for the inverse cotangent. Some consider the continuous range $(0, \pi)$ (Bourne 2011), whereas others use the discontinuous range $(-\pi/2, 0) \cup (0, \pi/2]$ (Weisstein 2011); these discussions go beyond the present work. My students learn to work both cases to keep an open perspective.

To find the inverse cotangent for the range $(0, \pi)$, known as inverse cotangent A, the upper-half unit-circle worksheet is used. A horizontal line is drawn tangent to the circle at (0, 1). This is the cotangents line with equation $y = 1$. The value of the argument is located at a point on the cotangents line, at a distance from (0, 1) equal to the given argument value. A ray is drawn from the origin to that point, forming an angle that can be measured with a protractor. Examples from classwork show that $\cot^{-1}(0.4) = \text{arc}\cot(0.4) \approx 68°$ and $\cot^{-1}(-1.3) = \text{arc}\cot(-1.3) \approx 142°$ (see **fig. 4a**).

For the inverse cotangent with discontinuous range $(-\pi/2, 0) \cup (0, \pi/2]$, the right-half unit-circle worksheet is used. In this case, two cotangents lines are needed: an upper line $y = 1$ and a lower line $y = -1$. The value of the given argument is located on the appropriate cotangents line, and the ray is drawn from the origin to that location. Examples show that $\cot^{-1}(0.4) = \text{arc}\cot(0.4) \approx 68°$ and $\cot^{-1}(-1.3) = \text{arc}\cot(-1.3) \approx -38°$ (see **fig. 4b**).

As expected, the procedures for the range $(0, \pi)$ and for the range $(-\pi/2, 0) \cup (0, \pi/2]$ give angle rays that are directly opposite each other when the argument is a negative number.

The Inverse Secant and Inverse Cosecant

To evaluate the inverse secant and cosecant, students are reminded that an angle's ray intersects the tangents line and cotangents line to form segments with lengths, measured from the origin, equal to the secant and cosecant values, respectively. Therefore, to find the inverse secant, one must use a segment of appropriate length (corresponding to the given value of the argument) contained in a ray from the origin intersecting the tangents line. To find the inverse cosecant, we follow

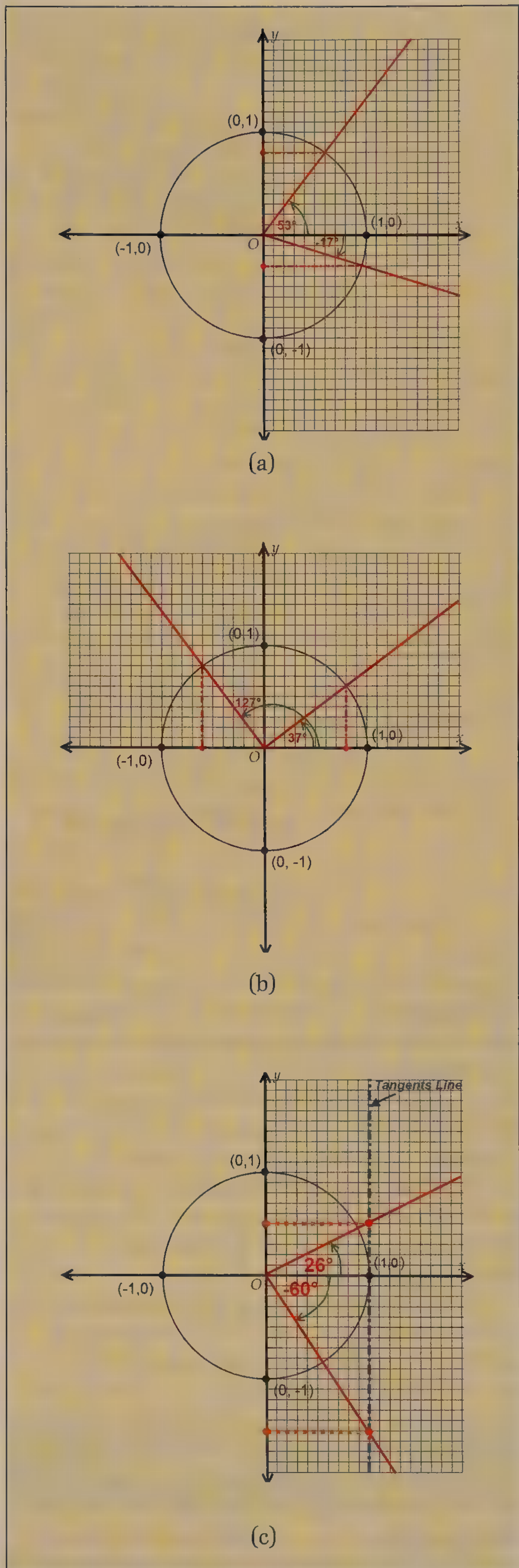


Fig. 3 Approximate values for the inverse sine (**a**), inverse cosine (**b**), and inverse tangent (**c**) can be measured with a protractor.

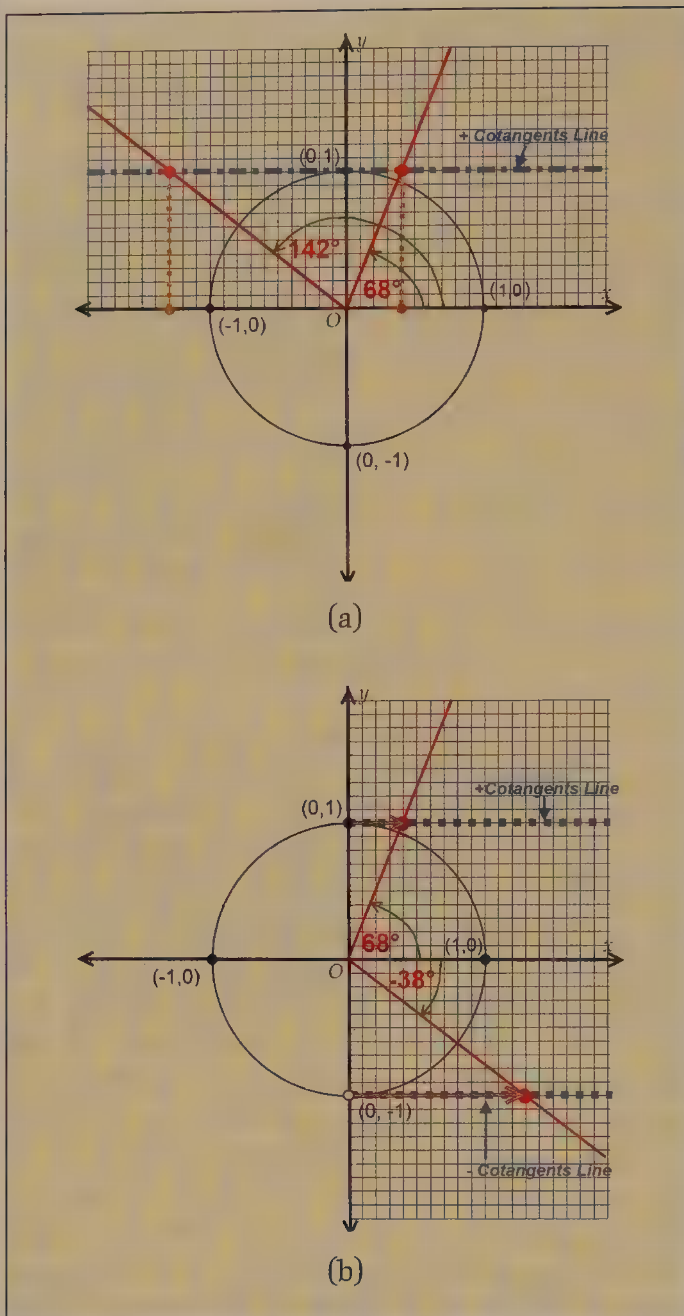


Fig. 4 Approximate values for the inverse cotangent may depend on the defined range (0, π) (**a**) or ($-\pi/2$, 0) ∪ (0, $\pi/2$] (**b**).

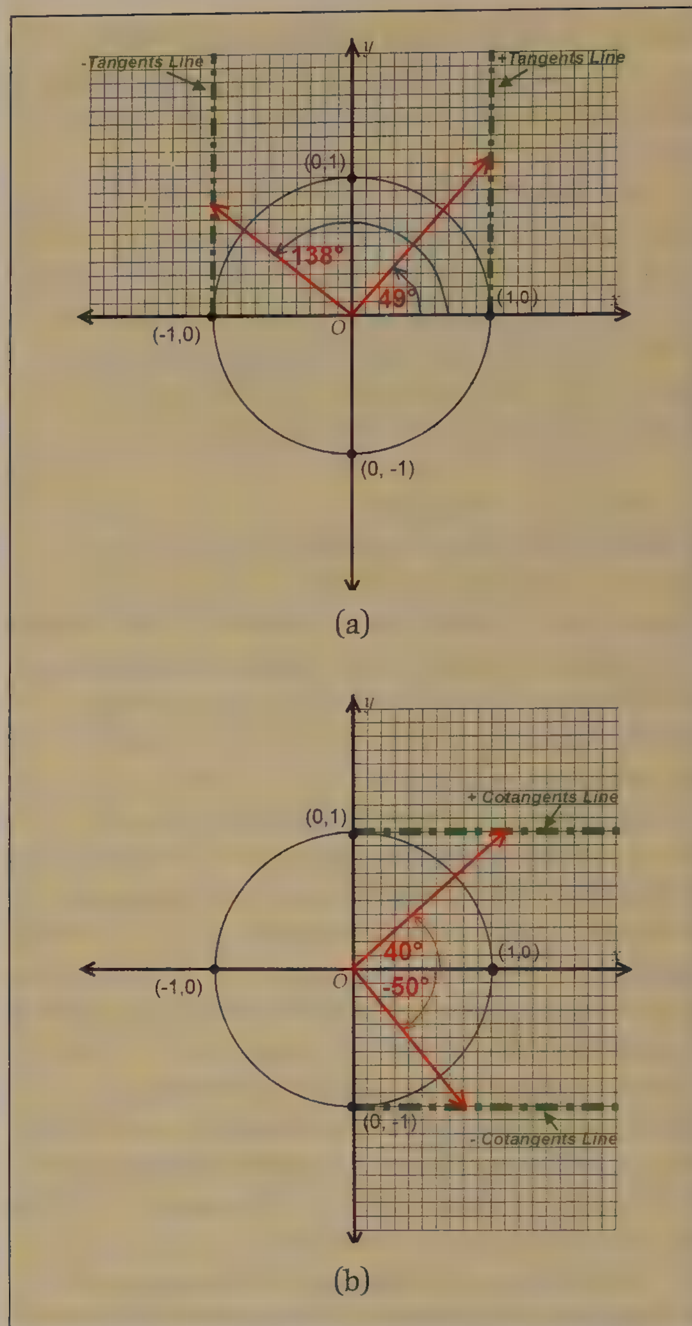


Fig. 5 Rays that intersect the tangents and cotangents lines create angles that represent the inverse secant (**a**) and the inverse cosecant (**b**).

the same process but intersect the cotangents line. The range of the inverse secant is $[0, \pi/2) \cup (\pi/2, \pi]$. Therefore, for the inverse secant, the upper half of the unit circle worksheet is used. The range of the inverse cosecant is $[-\pi/2, 0) \cup (0, \pi/2]$, so the right half of the worksheet is used.

To find the inverse secant, two vertical tangents lines are needed, $x = 1$ and $x = -1$, for positive and negative arguments, respectively. To find the inverse cosecant, two horizontal cotangents lines are needed, $y = 1$ and $y = -1$, for positive and negative arguments, respectively. A segment is drawn from the origin having a length corresponding to the value of the argument—for the inverse secant, the segment must intersect the tangents line; and for the inverse cosecant, the segment must intersect the cotangents line. Rays containing the segments define angles with respect to the positive x-axis, and the angle measures provide the output of the inverse function.

There are many ways to construct the ray of the angle. My students found a convenient way—they used a drafting compass to transport the length from the x-axis scale to intersect the tangents or cotangents lines. This construction can also be done using a ruler.

Examples from exercises worked in class show $\sec^{-1}(1.5) = \text{arc sec}(1.5) \approx 49°$, and $\sec^{-1}(-1.3) = \text{arc sec}(-1.3) \approx 138°$ (see **fig. 5a**); and $\csc^{-1}(1.5) = \text{arc csc}(1.5) \approx 40°$, and $\csc^{-1}(-1.3) = \text{arc csc}(-1.3) \approx -50°$ (see **fig. 5b**).

APPLICATION OF THE METHOD

As indicated in the introduction, most trigonometry textbooks show how to find the sine and cosine functions for a given angle from the intersecting point $P(x, y)$ of its ray on the unit circle, the cosine corresponding to x and the sine to y. The other

functions are found from ratios and reciprocals of these two. Few textbooks show how to use the unit circle to find all six trigonometric functions from the lengths formed by the angle's ray in the circle (see **opening art** and **fig. 1**). The reason why students do not know how to find all functions directly from the unit circle may be rooted in methods advocated in the last decades, as described by Kendal and Stacey (1996, 1998).

However, the determination of the values of inverse trigonometric functions using exclusively the unit circle has been left out altogether. A practical method has been described here to fill this gap. With this method, my students have improved their skills and have overcome their misconceptions about inverse trigonometric functions.

ACKNOWLEDGMENTS

I acknowledge the work and comments of my students of analytic trigonometry, in particular those who gave permission to use their exercises as examples. I also acknowledge the support of Florida State University Panama and colleagues.

BIBLIOGRAPHY

Aufmann, Richard N., Vernon C. Barker, and Richard D. Nation. 2008. *College Trigonometry*. 6th ed. Belmont, CA: Houghton-Miffin/Cengage Learning.

Barnett, Raymond A., Michael R. Ziegler, and Karl E. Byleen. 2011. *Analytic Trigonometry with Applications*. 11th ed. New York: John Wiley and Sons.

Bourne, Murray. 2011. "Which Is the Correct Graph for Arccot *x*?" SquareCircleZ: Interactive Mathematics. May 3, 2011. http://www.squarecirclez.com/blog/which-is-the-correct-graph-of-arccot-x/6009

Bressoud, David M. 2010. "Historical Reflections on Teaching Trigonometry." *Mathematics Teacher* 104 (2): 107–12.

Cohen, David, Theodore B. Lee, and David Sklar. 2005. *Precalculus with Unit Circle Trigonometry*. 4th ed. Belmont, CA: Cengage Learning.

Kendal, Margaret, and Kaye Stacey. 1996. "Trigonometry: Comparing Ratio and Unit Circle Methods." In *Technology in Mathematics Education*. Proceedings of the 19th Annual Conference of the Mathematics Education Research Group of Australasia (MERGA), edited by P. C. Clarkson, pp. 322–29. Melbourne: MERGA.

———. 1998. "Teaching Trigonometry." *The Australian Mathematics Teacher* 54 (1): 34–39.

Larson, Ron. 2011. *Trigonometry*. 8th ed. Belmont, CA: Brooks/Cole Cengage Learning.

Marolois, Samuel. 1627. *Opera Mathematica*. Amsterdam: Ian Ianssen. doi:http://dx.doi.org/10.3931/e-rara-1070

Moussa, Ali. 2011. "Mathematical Methods in Abu al-Wafa's Almagest and the Quibla Determination." *Arabic Sciences and Philosophy* 21 (1): 1–56. doi:http://dx.doi.org/10.1017/S095742391000007X

Stewart, James, Lothar Redlin, and Saleem Watson. 2012. *Precalculus: Mathematics for Calculus*. 6th ed. Belmont, CA: Brooks/Cole Cengage Learning.

Sullivan, Michael. 2012. *Trigonometry: A Unit Circle Approach*. 9th ed. New York: Pearson Education.

Swokowski, Earl W., and Jeffery A. Cole. 2009. *Algebra and Trigonometry with Analytic Trigonometry*. 12th ed. Belmont, CA: Cengage Learning.

Trigsted, Kirk. 2012. *Trigsted Trigonometry*. MyMathLab e-Course Series and e-Text Reference. New York: Pearson Education.

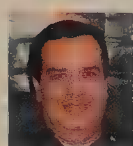
Van Brummelen, Glen. 2009. *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry*. Princeton, NJ: Princeton University Press.

Viète, François. 1579. *Canon Mathematicus Seu ad Triangula: Cum Appendicibus*. Joannem Mettayer Publisher. http://gallica.bnf.fr/ark:/12148/bpt6k52673b.r=Francois+Viete.langEN

Weisstein, Eric W. 2011. "Inverse Cotangent." *MathWorld*—A Wolfram Web Resource. http://mathworld.wolfram.com/InverseCotangent.html

Young, Cynthia Y. 2010. *Trigonometry*. 2nd ed. Hoboken, NJ: John Wiley and Sons.

Zill, Dennis G., and Jacqueline M. Dewar. 2012. *Trigonometry*. 3rd ed. Sudburg, MA: Jones and Bartlett Learning.



AZAEL BARRERA, abarrera@fsu.edu, teaches mathematics, physics, and astronomy at Florida State University Panama in the Republic of Panama. His interests include physics and mathematics education research in active learning and engagement methods. He has also served as technology director for the Science and Technology Secretary of Panama.

SEPTEMBER

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

This month's theme is "Thirty Days Hath September." The answers to this Calendar are the integers from 1 to 30. Each integer occurs exactly once.

1

Sam makes \$10.50 per hour. When he works a 40-hour week, his after-tax take-home pay is \$344.27. How large a raise—expressed as a percentage—would Sam need so that his new take-home pay equals his current gross pay? Assume that the tax *rate* does not change. (Today is Labor Day in the United States.)

2

Given the following system of equations—

$$\begin{aligned} 9x - 6y &= 21 \\ 6x - 4y &= k \end{aligned}$$

—find the value of k such that the system has an infinite number of solutions.

3

The side lengths of a triangle are the integral roots of the equation

$$x^3 - 16x^2 + 85x - 150 = 0.$$

What is the area of the triangle?

4

The area of an equilateral triangle is 2/3 the area of a regular hexagon. If the perimeter of the hexagon is 6, what is the perimeter of the triangle?

5

Points A (–17, 2), B (–8, –11), and C (14, –15) lie on circle O. Construct circle O. What is the length of the radius?

6

Given the following system of equations—

$$\begin{aligned} 2x + 8y &= 5 \\ cx + 36y &= 8 \end{aligned}$$

—find the value of c such that the system has no solutions.

7

We are given points T (8, 0) and M (13, 0). Find the sum of the coordinates of a point P such that P lies on the line $y = x/2 + 1$ and $\triangle PTM$ is an isosceles obtuse triangle.

8

The French priest, scientist, and mathematician Marin Mersenne was born on this date in 1588. He studied prime numbers of the form $2^p - 1$. If $2^p - 1$ is prime, then p is prime, but the converse is not true. Find the only prime p less than 20 for which $2^p - 1$ is not prime by finding the largest prime factor of 14,641.

9

Determine the number of ordered pairs of positive integers (x, y) that satisfy the following equation:

$$20x + 14y = 2014$$

10

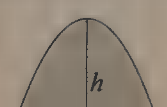
Find the value of b such that this system of equations has an infinite number of solutions:

$$\begin{aligned} -3x + 2y - 4z &= 6 \\ 7x + 6y + 4z &= b \\ -5x - 4y - 3z &= -1 \end{aligned}$$

11

Today's date, 9/11/14, can be viewed as a numerical expression instead of a date. Find the sum of the digits required to write this expression as a single proper fraction in lowest terms.

The formula for the area of a parabolic segment is given by $A = (2/3)hw$, where h and w are defined as in the diagram. If the area bounded by $y = -(1/2)x^2 + c$ and the x-axis is 144 units^2, what is the value of c?

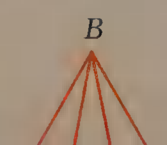


The word HIDING conceals a six-digit prime number. Each distinct letter represents a single digit. H, I, and D are prime, and the consecutive letter pairs HI, DI, and NG are two-digit primes. All permutations of an H and two Is are prime, while HIH is divisible by (2D + 1). If N = 2^I, find

Find the sum of the solutions to the equation

$$3(3^{2x}) - 28(3^x) = -9.$$

Equilateral $\triangle ABC$ has side length 12. Points X and Y trisect $\overline{AC}$, so triangles ABX, XBY, and YBC have equal area. But their perimeters are not equal. If $a - b\sqrt{c}$, written in simplified radical form, is the positive difference between the perimeter of $\triangle ABX$ and



The sum of three consecutive positive integers is 2.5% of the product of the same three integers. What is the value of the smallest of the three integers?

16

The number 121 is the only perfect square that can be written as the sum of consecutive powers beginning with 1. Find a such that

$$a^0 + a^1 + \cdots + a^4 = 121.$$

17

The prolific Swiss mathematician Leonhard Euler died on this date in 1783. He introduced what is called the Euler phi function, $\phi(n)$, which gives the number of positive integers less than n and relatively prime to n. For example, $\phi(12) = 4$ because 1, 5, 7, and 11 are relatively prime to 12. (The number 1 is counted as relatively prime to all integers.) Find $\phi(30)$.

18

How many subsets of $\{1, 2, 3, 4, 5\}$ contain at least one even integer?

19

A circle with radius $\sqrt{170}$ passes through the points (0, 2) and (2, 8). If the center of the circle lies in quadrant I, find the x-coordinate of the center of the circle.

20

A block of wood in the shape of a right rectangular prism has dimensions 1 ft. × 1 ft. × 2 ft. If Tom saws it in half to create two blocks with dimensions 1 ft. × 1 ft. × 1 ft., what is the percentage increase in total surface area?

21

How many zeros are needed to write the base-10 number

$$2^{2^{2^{2}}}$$

in base 2?

22

Three numbers form a geometric sequence. The arithmetic mean of the first two is 6, and the arithmetic mean of the second and third is 18. What is the largest term of this sequence?

23

The French mathematician Henri Brocard (1845–1922) posed the following question: For what values of n is $n! + 1$ a perfect square? Only three solutions are known; all three solutions have $n < 10$. Find the smallest solution.

24

If the volume of a cube is 48% greater than the volume of a second cube, by what percentage does the surface area of the first cube exceed the surface area of the second? (Round to the nearest integer.)

25

We have the following set of values:

11, 12, 17, 18, 23, 29, and 30

Removing one value causes the mean to decrease by 1.5. Find that value.

26

If the width of a certain rectangle is increased by 2 and its length is cut in half, its area remains unchanged. If the perimeter of the original rectangle is 40% greater than the perimeter of the altered rectangle, find the original perimeter.

27

Two legs of a right triangle are in the ratio 5 : 6; the hypotenuse has length $2\sqrt{61}$. If the longer leg is doubled and the shorter leg does not change, what is the length of the hypotenuse?

28

Find the number of ordered pairs of integers that satisfy the inequality

$$x^2 + y^2 \le 5.$$

29

A right triangle has legs $6\sqrt{13}$ and $4\sqrt{13}$. Find the distance between the midpoint of the hypotenuse and the foot of the altitude to the hypotenuse.

30

SOLUTIONS to calendar

Looking for more Calendar problems?
Visit www.nctm.org/publications/calendar/default.aspx?journal_id=2 for a collection of previously published problems—sortable by topic—from *Mathematics Teacher.*

Department editors
Margaret Coffey, *Margaret.Coffey@fcps.edu, Thomas Jefferson High School for Science and Technology, Alexandria, VA 22312; and* ***Art Kalish,*** *artkalish@verizon.net, Director of the Institute of MERIT at SUNY College at Old Westbury*

The Editorial Panel of *Mathematics Teacher* is considering sets of problems submitted by individuals, classes of prospective teachers, and mathematics clubs for publication in the monthly Calendar. Send problems to the Calendar editors. Remember to include a complete solution for each problem submitted.

Another source of problems in calendar form available from NCTM is the 100 Problem Poster (stock number 13207, $9.00). Individual members receive a 20 percent discount off this price. A catalog of educational materials is available at www.nctm.org.–Eds.

1. 22. We know that

$$\frac{\text{new take-home pay} - \text{current take-home pay}}{\text{current take-home pay}} = \text{percentage increase},$$

where new take-home pay = current gross pay. Therefore,

$$\frac{10.5(40) - 344.27}{344.27} = 22\%.$$

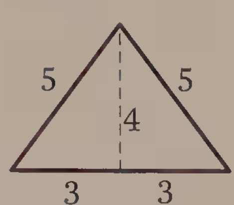
2. 14. The system will have an infinite number of solutions if the two equations represent coincident lines. Divide both sides of the first equation by 3, and divide both sides of the second by 2:

$$\begin{matrix} 3x - 2y = 7 \\ 3x - 2y = 0.5k \end{matrix} \rightarrow 0.5k = 7 \rightarrow k = 14$$

3. 12. The roots of the equation must be factors of 150. There are a dozen possibilities to check, but we will begin with smaller values since the sum of the roots is 16. The coefficient 85 suggests that 5 may be a root. Using synthetic substitution, we have the following:

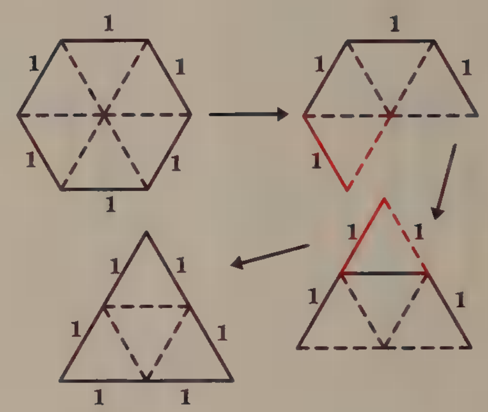
$$\begin{array}{c|cccc} & 1 & -16 & 85 & -150 \\ 5 & & 5 & -55 & 150 \\ \hline & 1 & -11 & 30 & 0 \end{array}$$

So we can rewrite the cubic polynomial with one linear factor and one quadratic factor: $(x - 5)(x^2 - 11x + 30) = 0 \rightarrow (x - 5)(x - 5)(x - 6) = 0$. We now find the area of an isosceles triangle with legs 5 and base 6. When we construct the altitude, we observe that the triangle has been divided into two 3-4-5 right triangles. The area is $3 \cdot 4 = 12$.



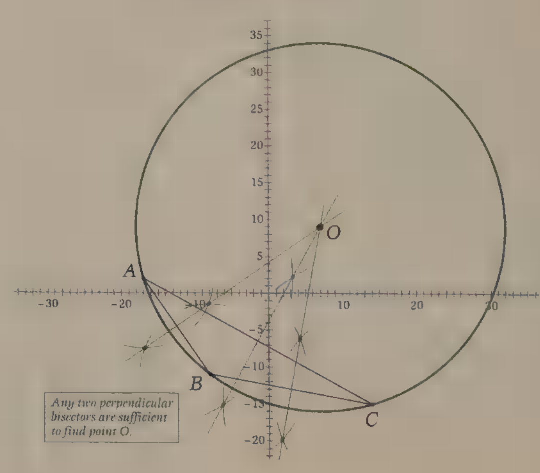
4. 6. A regular hexagon with perimeter 6 has side length 1. Use the 30-60-90° right-triangle relationship to find the apothem: $\sqrt{3}/2$. The area $A = ap/2 = (\sqrt{3}/2)(6/2) = 3\sqrt{3}/2$. So the triangle has area $\sqrt{3}$. The area of an equilateral triangle is $s^2\sqrt{3}/4$, where s is the length of a side. If $s^2\sqrt{3}/4 = \sqrt{3}$, then $s^2 = 4 \rightarrow s = 2$, and the perimeter of the triangle is 6.

Alternate solution: A simple "proof without words" gives a more elegant solution:



5. 25. To construct a circle through three given points, choose any two pairs of points and for each pair construct the segment that joins them. Construct the perpendicular bisector of each segment. The circle's center O (7, 9) is the intersection of the perpendicular bisectors. The radius OC can be found using the distance formula:

$$d = \sqrt{(7 - 14)^2 + \left(9 - (-15)\right)^2}$$
$$= \sqrt{49 + 576} = 25$$



6. 9. Since 36/8 = 4.5, we will multiply both sides of the first equation by 4.5:

$$9x + 36y = 22.5$$
$$cx + 36y = 8$$

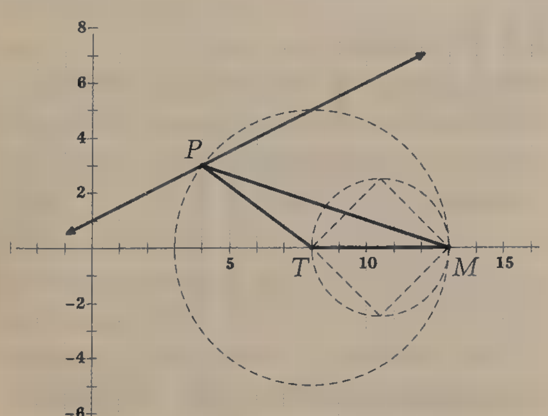
Subtract the second equation from the first to obtain $(9 - c)x = 14.5$. Since the right side is nonzero, the system will have no solution if the left side is 0. Therefore, the system will have no solution when $c = 9$.

7. 7. Construction offers a straightforward solution. If $\triangle TMP$ is isosceles, then $\overline{TM}$ is either the base or a leg. Graph the line $y = x/2 + 1$. Construct a circle with $\overline{TM}$ as the diameter. If $\overline{TM}$ is the base of $\triangle TMP$, then P must lie on the perpendicular bisector of $\overline{TM}$ within the constructed circle. But the given line does not intersect the circle, so $\overline{TM}$ must be a leg. Construct a circle with center T and radius TM. Triangle TMP is isosceles for all points P on the circle, but only $P(4, 3)$ and $P(8, 5)$ also lie on the graphed line. For $P(8, 5)$ the triangle is right, not obtuse, so the only solution is $P(4, 3)$. The sum of the coordinates is 7.

Note: If P were not a lattice point, then the construction would still give us useful information about the number of solutions and approximate location of the solutions. In that case, the exact solution would be found as follows:

$$d = \sqrt{(x-8)^2 + ((x/2)+1)^2} \rightarrow$$
$$25 = x^2 - 16x + 64 + x^2/4 + x + 1 \rightarrow$$
$$x = 4 \text{ or } x = 8$$

If $x = 4$, $y = 3$; and if $x = 8$, $y = 5$. Proceed as before.



8. 11. Some students may recognize 14,641 as 11^4. Otherwise, observe that the sum of the first, third, and fifth digits is 1 + 6 + 1 = 8, which equals the sum of the second and fourth digits: 4 + 4. So 11 is a factor of 14,641. The other factor is 1331, which is also divisible by 11. Since $1331/11 = 121 = 11^2$, the only prime factor of 14,641 is 11. Thus, p is 11. Notice that $2^{11} - 1 = 2047$ is composite; its prime factors are 23 and 89.

9. 15. This equation is equivalent to $10x + 7y = 1007$. The smallest possible value for y is 1; and when $y = 1$, $x = 100$. The product $7y$ must have the digit 7 in the ones place so that $1007 - 7y$ is divisible by 10. Therefore, y can take only the values from the set {1, 11, 21, ..., 131, 141}, for a total of fifteen positive, integral values. We have, therefore, fifteen ordered pairs of positive integers.

Alternate solution: Since the solutions must be pairs of positive integers, all solutions will lie in quadrant I. As before, the original equation is equivalent to $10x + 7y = 1007$, and the ordered pair (100, 1) gives the solution with the smallest y-value. Since the slope of the graph of the equation is –10/7, we can "follow" the line through (100, 1) to the left, moving 7 units to the left and 10 units up to reach the next integral solution. Since 100/7 = 14 2/7, we can find fourteen solutions in addition to (100, 1) before we reach the y-axis.

10. 2. We can use linear combinations to reduce the system of three equations in three unknowns to two equations in x and y only. Then we proceed in the same manner as we did in solving the problem for September 2. Adding the first and second equations gives us $4x + 8y = 6 + b$. Multiply the second equation by 3 and the third by 4:

$$21x + 18y + 12z = 3b$$
$$-20x - 16y - 12z = -4$$

Then add to obtain $x + 2y = 3b - 4$. The left side of this resulting equation multiplied by 4 will match the left side of our previous equation. If there are an infinite number of solutions, then the right sides must be equal; that is, $12b - 16 = 6 + b \rightarrow 11b = 22 \rightarrow b = 2$.

Alternate solution 1: The choice of linear combinations is certainly not unique. Multiply the first equation by 2 and add it to the third; this results in $-11x - 11z = 11 \rightarrow x + z = -1$. Multiply the second equation by 2 and add it to 3 times the third equation; this results in $-x - z = 2b - 3$. The two equations in x and z can be added; we obtain $0 = 2b - 4$. If $b = 2$, the equation is true, and the system will have an infinite number of solutions. If b has any value other than 2, then the equation is false, and the system will have no solutions.

Alternate solution 2: The augmented matrix representing the linear system,

$$\left[\begin{array}{ccc|c} -3 & 2 & -4 & 6 \\ 7 & 6 & 4 & b \\ -5 & -4 & -3 & -1 \end{array}\right],$$

can be put into row-reduced echelon form as

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & \frac{b}{16} - \frac{9}{8} \\ 0 & 1 & \frac{-1}{2} & \frac{3(b+14)}{32} \\ 0 & 0 & 0 & \frac{3b-6}{32} \end{array}\right].$$

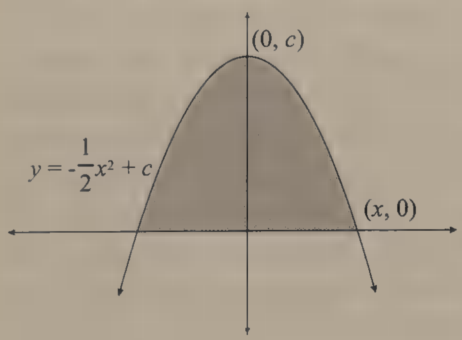
There will be an infinite number of solutions if the third row of the matrix represents the equation $0x + 0y + 0z = 0$. That is, if $(3b - 6)/32 = 0$, then $b = 2$.

Students should be familiar with the geometric interpretation of the September 2 problem. In this problem, we see equations in three variables. Each represents a plane in three-dimensional space. We obtain an infinite number of solutions when the three planes coincide or when the three places intersect in a single line.

11. 19. The expression 9/11/14 is the same as 9/11 · 1/14 = 9/154. The sum of the digits is 9 + 1 + 5 + 4 = 19.

12. 18. Given the equation $y = -(1/2)x^2 + c$, we note that $y = c$ when $x = 0$. So the y-intercept is c, and c will replace h in the area formula. The x-intercept is $(x, 0)$ or, equivalently, $(x, -(1/2)x^2 + c)$. Equating the values for the y-coordinate gives us

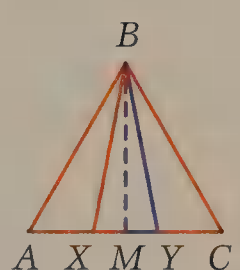
$-(1/2)x^2 + c = 0 \rightarrow (1/2)x^2 = c$. Replacing w in the area formula with $2x$, we have $A = (2/3)c \cdot 2x = (2/3)(x^2/2)(2x) = (2/3)x^3 = 144 \rightarrow x = 6$. Finally, $c = (1/2)6^2 = 18$.



13. 17. Since H, I, and D are prime, they must be elements of {2, 3, 5, 7}. Since all permutations of an H and two Is are prime, neither H nor I can be 2 or 5. The number HIH is either 737 or 373, and one of the two is divisible by (2D + 1), which can be 2(2) + 1 = 5 or 2(5) + 1 = 11. The letter D must be 5, and 737 is divisible by 11. We know that H = 7, I = 3, and D = 5. We also have $N = 2^I = 8$. Since NG is prime and G ≠ 3, we conclude that G = 9. Finally, H + I + D + I + N – G = 7 + 3 + 5 + 3 + 8 – 9 = 17. Note: The number 735,389 is one of a pair: 735,389 and 735,391 are twin primes.

14. 1. Writing 3^{2x} as $(3^x)^2$ helps show that the equation is quadratic in 3^x. We let $a = 3^x$ to write $3a^2 - 28a + 9 = 0$. Factoring gives $(3a - 1)(a - 9) = 0 \rightarrow a = 1/3$ or $a = 9$. Replacing a with 3^x gives us $3^x = 1/3 \rightarrow x = -1$ or $3^x = 9 \rightarrow x = 2$. The sum of the solutions is 1.

15. 23. The perimeter of $\triangle ABX$ is $AB + BX + AX$, and the perimeter of $\triangle XBY$ is $BX + BY + XY$. Since $AX = XY$, the positive difference of the perimeters must be $AB - BY$. Let M be the midpoint of $\overline{AC}$. Since $\overline{MB}$ is the altitude of $\triangle ABC$, its length is $6\sqrt{3}$. Apply the Pythagorean theorem to $\triangle MBY$ to find BY: $BY^2 = (6\sqrt{3})^2 + 2^2 = 112 \rightarrow BY = 4\sqrt{7}$. The positive difference is $a - b\sqrt{c} = 12 - 4\sqrt{7}$, so $a + b + c = 23$.



16. 10. Let n = the second of the three integers, and note that 2.5% is 1/40. Then $(n - 1) + n + (n + 1) = n(n^2 - 1)/40 \rightarrow 120n = n(n^2 - 1)$. Since $n \neq 0$, divide by n: $120 = n^2 - 1 \rightarrow n = 11$. Reject –11. If $n = 11$, the smallest of the three integers is 10.

17. 3. Since $4^4 = 256$, we know that $a < 4$. We try $3^0 + 3^1 + 3^2 + 3^3 + 3^4 = 1 + 3 + 9 + 27 + 81 = 121$.

18. 8. The positive integers less than 30 and relatively prime to 30 are 1, 7, 11, 13, 17, 19, 23, and 29. There are eight integers, so $\phi(30) = 8$.

Students may enjoy confirming that $\phi(30) = 8$ with the following identity. If p_1, p_2, p_3, … are the prime factors of n, then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\left(1 - \frac{1}{p_3}\right)\cdots.$$

19. 24. We could, of course, simply write out all the subsets, of which there are $2^5 = 32$, since the total is manageable, and count those that meet the condition. (Note that the count of 32 includes the empty set subset.)

A second approach might consider the phrase "at least one" in the problem statement and count the subsets that exclude both 2 and 4. Then subtract this number from 32. The number of subsets that exclude both 2 and 4 is the same as the number of subsets that can be formed from {1, 3, 5}, which is $2^3 = 8$. Then our answer is 32 – 8 = 24.

A third approach counts the number of subsets that contain one, two, three, four, or five elements. There are exactly 2 subsets that contain one element and that satisfy the condition—namely, {2} and {4}. To count the number of subsets that contain two elements and that satisfy the condition, we count all two-element subsets and subtract the subsets made up of 1, 3, 5 only: ${}_5C_2 - {}_3C_2 = 10 - 3 = 7$. Use the same reasoning for three-element subsets: ${}_5C_3 - {}_3C_3 = 10 - 1 = 9$. All 5 four-element subsets contain must contain at least one of {2, 4}, and the single five-element subset must be counted, too. We have 2 + 7 + 9 + 5 + 1 = 24. Although this method may be tedious, it gives students good combinatorics practice.

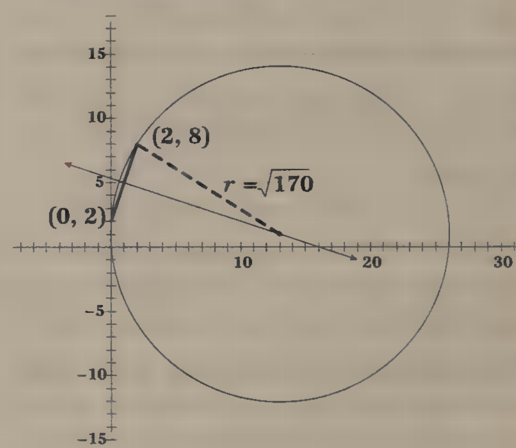
20. 13. The centers of *all* circles that contain (0, 2) and (2, 8) will lie on the perpendicular bisector of the segment with endpoints (0, 2) and (2, 8). That segment has midpoint (1, 5) and slope 6/2 = 3. Therefore, the perpendicular bisector has slope –1/3. An equation in point-slope form is $(y - 5) = (-1/3) \cdot (x - 1)$, which is equivalent to $y = -x/3 + 16/3$. So we can write the coordinates of the center as $(x, -x/3 + 16/3)$. The radius $r = \sqrt{170}$ is the distance from (0, 2) to the center. Use the distance formula as follows:

$$170 = (x - 0)^2 + \left(-\frac{x}{3} + \frac{16}{3} - 2\right)^2 \rightarrow$$

$$x^2 + \frac{x^2}{9} - \frac{20x}{9} + \frac{100}{9} - 170 = 0 \rightarrow$$

$$x^2 - 2x - 143 = 0$$

Factoring gives us $(x - 13)(x + 11) = 0 \rightarrow x = 13$ or $x = -11$. There are two circles with radius $\sqrt{170}$ that contain the two given points. We use $x = 13$, since we are told that the center lies in quadrant I.



21. 20. The surface area of the block of wood before sawing was $4(1 \cdot 2) + 2(1^2) = 10$ ft.^{2}. After sawing, the combined surface area is $2 \cdot 6(1^2) = 12$ ft.^{2}. Since the original surface area was 10 ft.^{2}, the increase of 2 ft.^{2} is a 2/10 = 20% increase.

22. 16. Observe first that $2^1 = 2_{10}$ is written 10_2 with one zero; $2^2 = 4_{10}$ is written 100_2 with two zeros. In general, 2^n in base 10 is written in base 2 as a 1 followed by n zeros. Now

$$2^{2^{2^2}} = 2^{2^4} = 2^{16},$$

so the base-2 representation will have sixteen zeros.

23. 27. Let the smallest term be a. Then the second and third terms are ar and ar^2, where r is the common ratio. We know that $(a + ar)/2 = 6 \rightarrow a(1 + r) = 12$ and that $(ar + ar^2)/2 = 18 \rightarrow ar(1 + r) = 36$. Use the division property of equality (because $r \neq -1$) to write

$$\frac{ar(1+r)}{a(1+r)} = \frac{36}{12} \rightarrow r = 3.$$

Therefore, $a(1 + 3) = 12 \rightarrow a = 3$. The second term must be $3 \cdot 3 = 9$, and the largest term in the sequence is $3 \cdot 9 = 27$.

24. 4. Since all three known solutions are less than 10, we can find all of them fairly easily. Our search results are summarized in the table below.

$n! + 1$	Perfect square?
$1! + 1 = 2$	no
$2! + 1 = 3$	no
$3! + 1 = 7$	no
$4! + 1 = 25$	5^2
$5! + 1 = 121$	11^2
$6! + 1 = 721$	no
$7! + 1 = 5041$	71^2

The smallest solution is $n = 4$.

25. 30. If two figures are similar with scale factor $a:b$, the ratio of their surface areas is $a^2 : b^2$, and the ratio of their volumes is $a^3 : b^3$. The ratio of volumes of the two cubes is $1:1.48$, so the scale factor is $1^{(1/3)} : (1.48)^{(1/3)}$, and the ratio of surface areas is $1^{(2/3)} : (1.48)^{(2/3)}$, or approximately $1:1.2987$. The surface area of the larger cube exceeds that of the smaller by about 30%, rounded to the nearest integer.

Alternate solution: Let c = the edge length of the smaller cube and $(c + a)$ = the edge length of the larger cube. Then $(c + a)^3 = 1.48c^3 \rightarrow (c + a)/c = \sqrt[3]{1.48} \rightarrow (c + a)/c \approx 1.1396 \rightarrow a \approx 0.1396c$. The surface area of the smaller cube is $6c^2$; the surface area of the larger cube is $6(c + a)^2 = 6(c + 0.1396c)^2 = 6(1.1396c)^2 \approx 1.1396^2 \cdot 6c^2$. Since $1.1396^2 \approx 1.30$, the surface area of the larger cube is approximately 30% greater than the surface area of the smaller. Note that we could have compared the areas of one face instead of the total surface area of the two cubes.

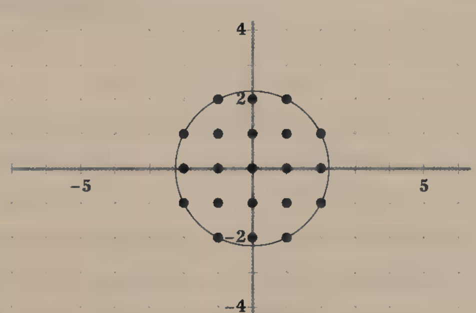
26. 29. The sum of the original set of numbers is 140, and the mean is $140/7 = 20$. The mean of the reduced set of numbers is $20 - 1.5 = 18.5$, so the sum of the six remaining numbers is $6(18.5) = 111$. Since $140 - 111 = 29$, the number removed must be 29.

Alternate solution: Let S be the sum of the seven values, and let a be the value to be subtracted. Then $(S - a)/6 = S/7 - 1.5$, so $a = S/7 + 9$. That is, the subtracted number is 9 more than the original mean. We find that the original mean is $140/7 = 20$, so $a = 29$. Confirm the solution: $140 - 29 = 111$, and $111/6 = 18.5$, which is 1.5 less than 20.

27. 28. Let w = the width of the original rectangle and let l = its length. Then $(w + 2)(l/2) = lw \rightarrow lw/2 + l = lw \rightarrow 2l = lw$. We can divide both sides by l, since $l > 0$, and we find that $w = 2$. Write an equation that relates the semiperimeters: $(l + 2) = 1.4(l/2 + 4) \rightarrow 0.3l = 3.6 \rightarrow l = 12$. The original perimeter is 28.

28. 26. Since $\sqrt{5^2 + 6^2} = \sqrt{61}$, which is half the given hypotenuse, we know that the original right triangle has legs 10 and 12. If we double the longer leg, we have legs 10 and 24, which means that the hypotenuse is $\sqrt{10^2 + 24^2} = 26$.

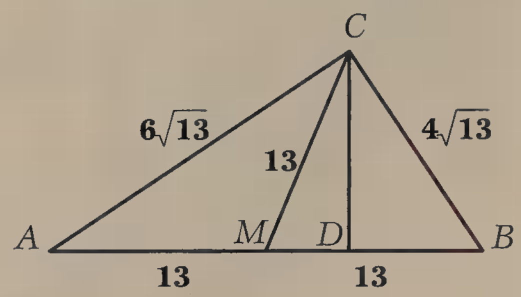
29. 21. If $x, y \in \mathbb{R}$, then the relation $x^2 + y^2 = 5$ is a circle of radius $\sqrt{5}$ centered at the origin. The circle passes through eight ordered pairs of integers: $(\pm1, \pm2)$ and $(\pm2, \pm1)$. Count the ordered pairs of integers that lie in the interior of the circle: $(0, 0)$, $(\pm1, \pm1)$, $(0, \pm1)$, $(\pm1, 0)$, $(0, \pm2)$, and $(\pm2, 0)$ for thirteen additional ordered pairs.



30. 5. Use the Pythagorean theorem to find hypotenuse:

$$\left(6\sqrt{13}\right)^2 + \left(4\sqrt{13}\right)^2 = 52 \cdot 13,$$

so the hypotenuse has length 26. Recall that the midpoint of the hypotenuse is equidistant from the three vertices, so we know that $MA = MB = MC = 13$ (see the fig.). Calculating the area of the triangle in two different ways results in $AC \cdot BC = CD \cdot AB \rightarrow (6\sqrt{13}) \cdot (4\sqrt{13}) = CD \cdot 26 \rightarrow CD = 12$. Finally, $MD = 5$ by the Pythagorean theorem.



Pr

obability & Perception

The Representativeness Heuristic in Action

Events that seem more representative may be judged more probable, so experiments and proof are needed to help students analyze a mathematical outcome.

Yun Lu, Francis J. Vasko, Trevor J. Drummond, and Lisa E. Vasko

When introducing the concept of probability to students, teachers may find that the following question stimulates lively and productive discussion:

> A die is rolled 20 times out of the view of an observer. Which result is more likely?
> (*a*) 11111111111111111111
> (*b*) 66234441536125563152

To a mathematician, it may be obvious that the two results are equally likely—that is, both have the same probability of occurring. However, for students who are encountering the concept of probability for the first time, it may not be obvious. In fact, this question was answered incorrectly in the Ask Marilyn column, which declared the answer to be (*b*) because "it was far more likely to be … a jumble of numbers" (vos Savant 2011).

The belief that a sequence such as 11111111111111111111 is less probable than a sequence such as 66234441536125563152 is often referred to as the *representativeness heuristic* (Kahneman and Tversky 1972; Shaughnessy 1977, 1992). According to these and other researchers, representativeness errors occur because people view one outcome—in this case, sequence (*b*): 66234441536125563152—as more representative of the population of "random" outcomes than another—in this case, sequence (*a*): 11111111111111111111. When one outcome is judged to be more representative of the population, it is consequently judged to be more likely.

FORMAL ANALYSIS

However, the problem is not so straightforward after all. Taking a closer look at the problem and carefully evaluating vos Savant's response, we see that the key reason for the confusion is that she refers to sequence (*b*) as a "jumble" of numbers. If sequence (*b*) is, as semantically stated in the problem, a fixed sequence of numbers, then both sequences have the same probability of occurring. However, if the probability of specifically obtaining a sequence of twenty 1s (sequence [*a*]) is being compared with the probability of obtaining some sequence where the numbers that occur may be in any order, then certainly the "jumbled" version is more likely. Specifically, the number of distinguishable ways of permuting the digits in sequence (*b*) is $(20!)/(4!3!3!3!3!4!) = 3.25909584 \times 10^{12}$. Hence, a semantically "jumbled" sequence is $3.25909584 \times 10^{12}$ times more likely to occur than any single "fixed" sequence.

There are various ways to try to prove the correct answer to prospective students of probability; the particular approach used depends on the students' mathematical background. If students understand the basic concepts of probability, such as independence and the multiplication rule of probability, and are comfortable with proof techniques, then we may provide them with a formal mathematical proof that any two sequences resulting from 20 rolls of a fair die are equally likely.

However, students may more easily understand the proof if it can be presented in a more intuitive and less rigorous manner. The outcome of each roll is simply one of six possible outcomes, so the probability of each outcome of one roll is 1/6. If we assume that the die is "fair" (that no side of the die holds a bias by weight or some other means), the outcome of each roll can be viewed as independent of all previous outcomes. Then, by the multiplication rule of probability, the probability of n rolls that occur in sequence can be obtained by multiplying the probability of each roll, which is $(1/6)^n$.

Hence, the probability of rolling sequence (*a*) is $(1/6)^{20}$, and the probability of rolling sequence (*b*) is also $(1/6)^{20}$; each sequence has the same probability of occurring. However, if the numbers in sequence (*b*) are allowed to be "jumbled"—that is, to be interchangeable with any other ordering of the same twenty digits—then the "jumbled" variations of sequence (*b*) are $3.25909584 \times 10^{12}$ times more likely to occur than a "fixed," or specific, sequence (*b*) or sequence (*a*).

CLASSROOM ACTIVITIES

If the prospective students of probability lack a background in mathematical proofs, then this reasoning may not elucidate the solution in a meaningful way. Consequently, we must seek alternative methods to help students learn to analyze the problem correctly. Hands-on classroom activities may work well in this situation. For example, students may physically roll a die twice to count and compare the frequency of the sequences. Tools such as graphing calculators or Microsoft Excel® spreadsheets may be used to simulate the process of rolling a die a large number of times.

We conducted a classroom activity to investigate equally likely probability in a class of twenty-four students who were college freshman business majors. The majority had not taken a proof-centric course and had no knowledge of probability other than their cumulative education (elementary school through high school).

The eighty-minute class session was divided into four parts: (1) a preactivity survey; (2) probability simulation activities; (3) a postactivity survey; and (4) a teacher-led discussion.

The preactivity survey consisted of a multiple-choice problem, asked seven times for $n = 2, 3, 4, 5, 6, 10, 20$:

> If you roll a die n times out of view, which result is more likely?
> (*a*) A sequence of all 1s
> (*b*) A fixed sequence of different numbers from 1 to 6
> (*c*) Equally likely

Students then were expected to defend their choices. One student chose answer (*a*) but did not give a reason. Sixteen students chose answer (*b*), whereas seven students chose (*c*), the correct answer.

Some typical reasons for choosing answer (*b*) follow:

- "It is hard to roll the same numbers n times in a row."
- "I picked the answers with different numbers because the chances of rolling all 1s are slim."
- ". . . it's very uncommon to roll the same exact number that many numbers in a row."

Table 1 Student Responses

	Number of Students			Percentage of Students	
Answer Choice	Preactivity Survey	Postactivity Survey		Preactivity Survey	Postactivity Survey
(*a*)	1	0		4.17%	0.00%
(*b*)	16	10		66.67%	41.67%
(*c*)	7	14		29.16%	58.33%

Some of the reasons given for choosing answer (*c*) included these:

- "There's the same probability to get the same number as to get other specific numbers in a specific order."
- "Because all six numbers on each die have the same chance."
- "There is an equal prob. [probability] for all of the possibilities because each number has an equal chance of being rolled; the chance for each digit is a 1 out of 6 chance."

Next, the class worked in groups of two, completing simulations of a die roll experiment to investigate the question asked on the survey. Specifically, each group physically rolled a die twice, used a graphing calculator simulation function, and applied an MS Excel spreadsheet simulation function (provided by the teacher).

The probability simulation activities included five experiments. The first activity prompted the class to physically roll a die twice for 120 times, record the outcomes for each trial, and compare the frequency of each sequence. We used 120 trials primarily because of time concerns; we needed to complete several activities within the limited class period. The second and third activities were designed to use the simulation feature of a TI-84 Plus™ graphing calculator to simulate n rolls of a die for $n = 3$ and $n = 4$. Students used the calculator function randInt(1,6,n) under the PRB option of the MATH menu to generate n random integers ranging from 1 through 6, thus simulating one trial of a die rolled n times. Students worked on the simulations for 120 trials for each experiment, recorded their data, and then compared the frequency of each sequence.

The last two activities were designed to use an MS Excel spreadsheet with a teacher-provided macro to simulate 5 and 6 rolls of a die for 100,000 trials. The MS Excel spreadsheet file (go to **www.nctm.org/mt059**) was provided ahead of time and available for download so that students could access it and perform the computations in a computer lab. The spreadsheet's macro automatically returns the total number of times each

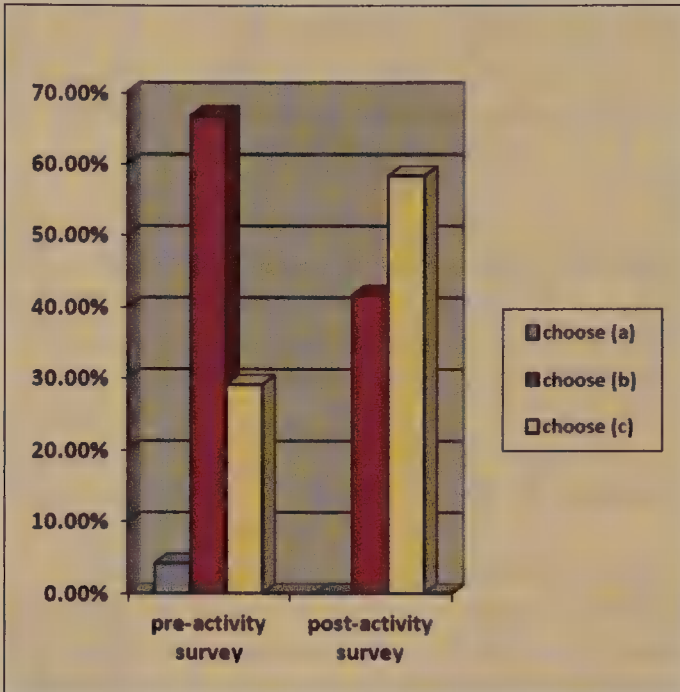


Fig. 1 Many students but not all showed improved mathematical reasoning after the activity.

specified sequence occurred within 100,000 trials. Specified sequences were as follows: a sequence of all 1s and a fixed sequence of numbers ranging from 1 to 6. Students repeated the process 60 times ($60 \times 100{,}000 = 6$ million total trials), recorded the outcomes, and compared the frequency of each of these sequences.

After all the experiments were completed, groups analyzed the results from each experiment. Students then completed the postactivity survey, which consisted of the same question as the preactivity survey. Having now performed the classroom activities, the students were expected to defend either maintaining or changing their original answers.

After the surveys had been collected, the class gathered together, and each group shared its observations and conclusions with the entire class. The teacher then led the whole class in a discussion to summarize underlying principles and conclusions. The students enjoyed the opportunity to explore a simple probability example (in this case, die-roll simulations) and analyze data on the basis of their own observations. Discussion among the groups facilitated sharing of ideas, so that the whole class came to agreement on the correct conclusion.

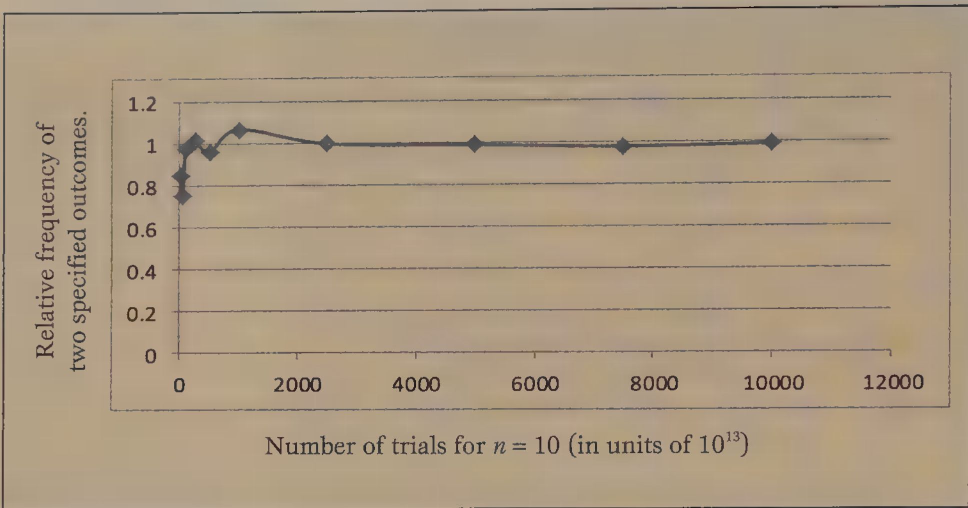


Fig. 2 As the number of trials increases, the relative frequency of two specified outcomes approaches 1.

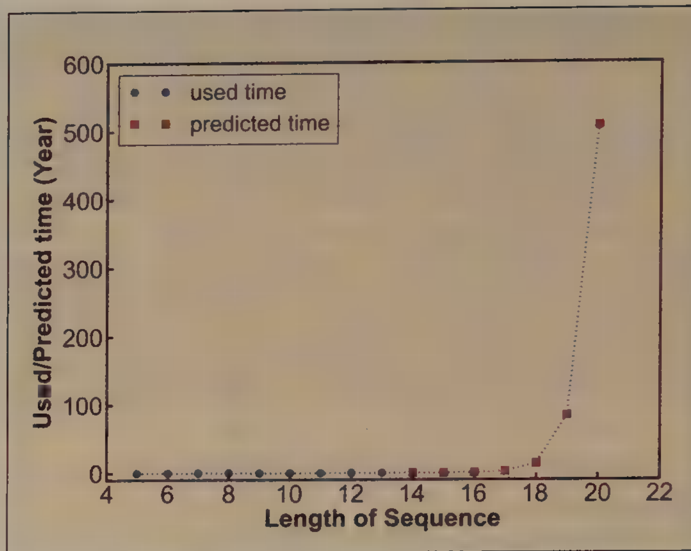


Fig. 3 Simulations of 100 • 6^n trials of generating a sequence of length n require calculation times that increase exponentially.

Table 1 and its corresponding figure (see **fig. 1**) present the results from the survey question:

> If I rolled a die 10 times out of your view, which result is more likely to occur?
> (a) 1111111111
> (b) 6125563152
> (c) Equally likely

Both survey results were collected before the class discussions. Although 41 percent of the students still chose (b) in the postactivity survey, the whole class appeared to agree on the correct solution (c) after the teacher-led discussion.

Some of the students who changed their answer from (b) to (c) gave these reasons:

- "Gathered from our results, it is the same outcome."
- "Equally likely because there's an equal chance to roll these numbers based on the results of our trials."
- "Each die has an equally likely chance of being rolled to the specific digits because each digit has the same probability of being rolled."

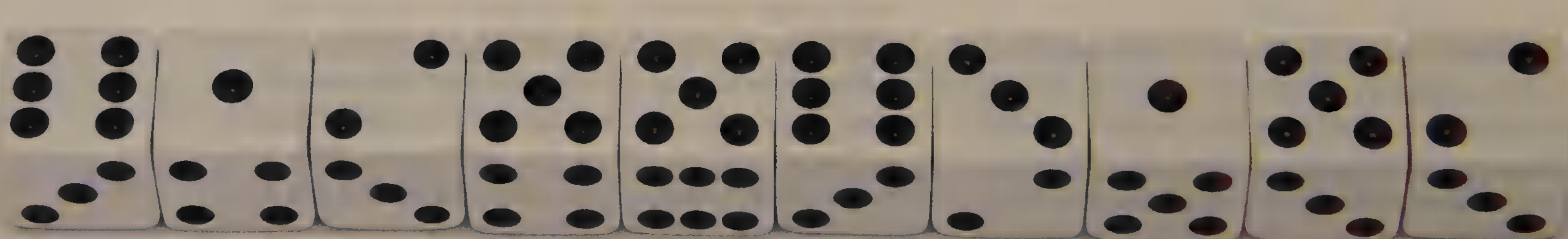
Overall, 58.33 percent of the students chose the correct answer (c) in the postactivity survey, compared with only 29.16 percent in the preactivity survey. This classroom activity generated a positive shift in the students' understanding of equally likely probability. Students also appreciated the opportunity to explore die-roll simulations and analyze the data on the basis of their own observations.

COMPUTATIONAL EFFORT

If the number of die rolls in each trial is a large integer n, then we expect to need a very large number of trials, 6^n, to achieve a specified outcome. Performing even one simulation may take too much time for students. Thus, we developed technology to simulate the experiment of rolling a die n times for 6^n trials repeated on one hundred computers simultaneously and counted the number of occurrences of two specified sequences of the length n.

For example, let $n = 10$ with sequence (a) as 1111111111 and sequence (b) as 6623444153. We may then generate the ratio of frequencies of sequence (a) to the frequencies of sequence (b). From **figure 2**, where the x-axis denotes the number of trials in units of 10^{13}, we can see that as the number of trials grows, the ratio of frequencies of the two specific sequences converges to 1.

Two observations with respect to the simulation are worth noting. First, the number of trials must be large enough to ensure that the specific sequences of length n can reasonably be expected to occur. For example, for $n = 10$, there are 6^{10} sequences possible, so if we repeat the experiment of simulating 10 rolls of a die for 6^{10} trials, we expect to see a specific sequence of length 10 to occur once. If we repeat the experiment 100 times, we expect to see 100 occurrences of each specific sequence of length 10. Second,



the running time increases exponentially with the length of the sequence generated.

The graph shown in **figure 3** predicts the running time used for simulation of $100 \cdot 6^n$ trials of generating sequence length n. For $n = 20$, simulating $100 \cdot 6^{20}$ trials of rolling a die 20 times, the processing speed of CPUs (specification: Intel® Xeon® CPU E5440 @ 2.83GHz), leads us to estimate that it would take more than 506 years to complete the calculations. We used the linear least-squares fitting method to calculate the simulation time.

A STRONG INFLUENCE TO OVERCOME

The student response to this probability problem clearly illustrates how strongly the representativeness heuristic can influence decision making by students who are unfamiliar with formal probability theory. Specifically, the process of estimating likelihoods for events according to how well an outcome is perceived to represent some aspect of its parent population is referred to as and defines the representativeness heuristic (Kahneman and Tversky 1972). When a college freshmen-level business-mathematics class with no assumed background in formal probability theory considered the original problem of a die rolled 20 times, initially 67 percent of the class believed that sequence (*b*) was more likely. To illustrate the powerful influence the representativeness heuristic can have on one's perception, even after the students performed a number of hands-on classroom activities aimed at empirically demonstrating that both sequences have the same probability, 42 percent of the students continued to believe sequence (*b*) to be more likely than sequence (*a*).

It is no surprise, then, that the power of the representative heuristic as a psychological construct supersedes rigorous mathematical definition (whether rigorously understood or not). In fact, the permanence of the heuristic—whereby students continue to exhibit the misunderstanding even in light of concrete, contradictory experiences—has been discussed by many (Konold et al. 1993; Shaughnessy 2003; Jones 2007).

As mathematics teachers, we must be aware that psychological factors sometimes influence students' mathematical understanding. Sometimes perception really does distort reality.

REFERENCES

Jones, Graham A. 2007. "Research in Probability: Responding to Classroom Realities." In *Second Handbook of Research on Mathematics Teaching and Learning*, edited by Frank K. Lester, pp. 909–56. Reston, VA: National Council of Teachers of Mathematics.

Kahneman, Daniel, and Amos Tversky. 1972. "Subjective Probability: A Judgment of Representativeness." *Cognitive Psychology* 3 (3): 430–54. http://dx.doi.org/10.1016/0010-0285(72)90016-3

Konold, Clifford, Alexander Pollatsek, Arnold Well, Jill Lohmeier, and Abigail Lipson. 1993. "Inconsistencies in Students' Reasoning about Probability." *Journal for Research in Mathematics Education* 24 (5): 392–414. http://dx.doi.org/10.2307/749150

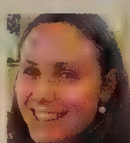
Shaughnessy, J. Michael. 1977. "Misconceptions of Probability: An Experiment with a Small-Group, Activity-Based, Model-Building Approach to Introductory Probability at the College Level." *Educational Studies in Mathematics* 8 (3): 295–316. doi: http://dx.doi.org/10.1007/BF00385927

———. 1992. "Research in Probability and Statistics: Reflections and Directions." In *Handbook of Research on Mathematics Teaching and Learning*, edited by Douglas A. Grouws, pp. 465–94. New York: Macmillan.

———. 2003. "Research on Students' Understanding of Probability." In *A Research Companion to Principles and Standards for School Mathematics*, edited by Jeremy Kilpatrick, W. Gary Martin, and Deborah Schifter, pp. 216–26. Reston, VA: National Council of Teachers of Mathematics.

vos Savant, Marilyn. 2011. "Ask Marilyn." *Parade Magazine*, October 23.



YUN LU, lu@kutztown.edu, is an associate professor of mathematics at Kutztown University of Pennsylvania. Her primary interests are mathematical logic, graph theory, and mathematics education. **FRANCIS J. VASKO,** vasko@kutztown, is a professor in the Department of Mathematics at Kutztown University of Pennsylvania. His primary interest is using mathematical optimization techniques and metaheuristics to mathematically formulate and solve real-world problems. **TREVOR J. DRUMMOND** received a B.S. in electrical engineering from Lehigh University in Bethlehem, Pennsylvania, and is pursuing an M.S. in electrical engineering from Drexel University in Philadelphia. **LISA E. VASKO** received a B.S. in mathematics and a M.S. in statistics, both from Lehigh University.

more4U

For an Excel file for Five Die Toss, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt059**.

Interpolation Polynomial

Two points determine a line. Three noncollinear points determine a quadratic function. Four points that do not lie on a lower-degree polynomial curve determine a cubic function.

In general, $n + 1$ points uniquely determine a polynomial of degree n, presuming that they do not fall onto a polynomial of lower degree. The process of finding such a polynomial is called *interpolation*, and the two most important approaches used are Newton's and Lagrange's interpolating formulas. Each has its advantages and disadvantages, as we will discuss. In this article, we show how both approaches can be introduced and developed at the precalculus level in the context of fitting polynomials to data. These methods bring some of the most powerful and useful tools of numerical analysis to the attention of students who are still at the introductory level while building on and reinforcing many fundamental ideas in algebra and precalculus mathematics.

PRECALCULUS MATHEMATICS

In algebra and precalculus, we emphasize the connection between the real zeros of a polynomial and its linear factors. For example, if $f(x) = x^2 - 5x + 6 = (x - 2)(x - 3)$, then $f(x)$ has two real zeros, $x = 2$ and $x = 3$, corresponding to the linear factors $x - 2$ and $x - 3$, respectively. Also, we connect a real zero with

and Curve Fitting

Two methods for precalculus students to find exact-fit polynomials.

Yajun Yang and Sheldon P. Gordon

an x-intercept. Conversely, if a parabola is given with two known x-intercepts, x_1 and x_2, then its equation must be of the form $f(x) = k(x - x_1)(x - x_2)$, where k is a constant that can be determined from additional information about the parabola.

For example, suppose that a parabola has x-intercepts at $x = 6$ and $x = -2$ and passes through the point (3, 20) (see **fig. 1**). Then we know that $f(x) = k(x - 6)(x + 2)$. To determine k, we use the given point (3, 20) to get $f(3) = k(3 - 6)(3 + 2) = 20$ and find that

$$k = \frac{20}{(3-6)(3+2)} = -\frac{4}{3}.$$

To see how the information about the parabola is used in the formula of the quadratic function, we express the function as

$$\begin{aligned} f(x) &= k(x-6)(x+2) \\ &= \frac{20}{(3-6)(3+2)}(x-6)(x+2) \\ &= 20 \cdot \frac{(x-6)(x+2)}{(3-6)(3+2)}. \end{aligned}$$

In a more general situation, if we have any three noncollinear points such as (1, 2), (3, 8), and (6, 4)

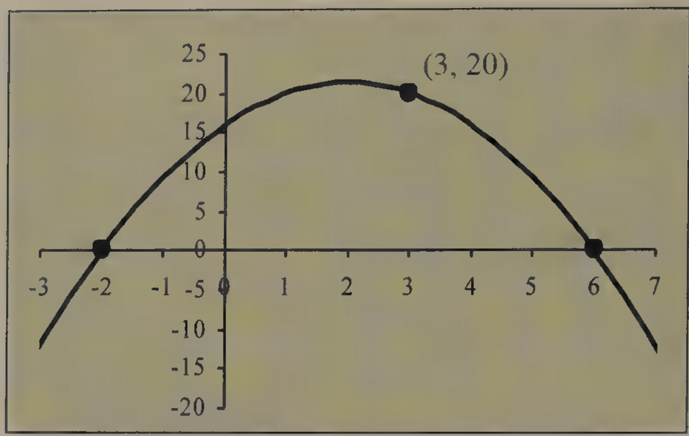


Fig. 1 A parabola may be defined by two x-intercepts and one additional point.

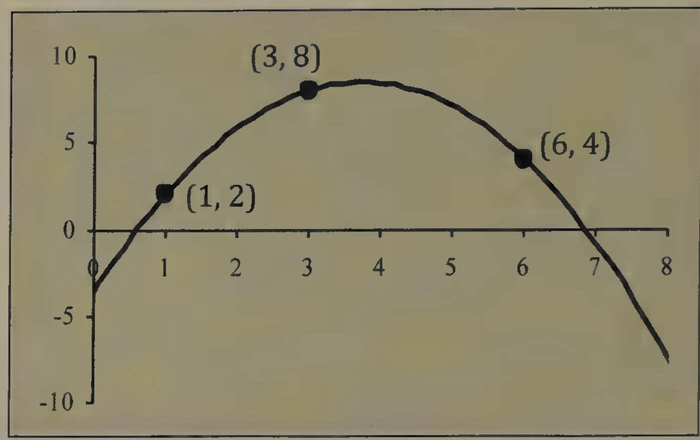


Fig. 2 A parabola may be defined by three noncollinear points.

on a parabola whose equation is $y = ax^2 + bx + c$, the points produce the system of linear equations:

$$\begin{aligned} a + b + c &= 2 \\ 9a + 3b + c &= 8 \\ 36a + 6b + c &= 4 \end{aligned}$$

Using either the substitution or the elimination method, we find that

$$a = -\frac{13}{15},\ b = \frac{97}{15},\ \text{and } c = -\frac{18}{5}.$$

So the quadratic function is

$$f(x) = -\frac{13}{15}x^2 + \frac{97}{15}x - \frac{18}{5}.$$

Its graph is shown in **figure 2** along with the interpolating points.

This approach is straightforward, but it has limitations if we want to extend it to more than three interpolating points. To fit a polynomial to $n + 1$ points, we would have to solve a system of $n + 1$ linear equations in $n + 1$ unknowns. Is there another way to find such an interpolating polynomial without involving heavy computations? Let's revisit the case in which we found the quadratic polynomial given the two x-intercepts, x_1 and x_2, and a point (p, q) on the parabola. If we repeat the derivation used with these parameters, we obtain the quadratic function

$$f(x) = q\frac{(x - x_1)(x - x_2)}{(p - x_1)(p - x_2)}. \qquad (1)$$

It is important to note that $f(x_1) = 0$, $f(x_2) = 0$, and $f(p) = q$. This observation holds the key to new ways of determining the quadratic function that passes through all three given points. It also provides the specific insight needed to extend the process to more than three points.

NEWTON'S INTERPOLATION

We know that any two points (x_0, y_0) and (x_1, y_1) determine a line. For the moment, suppose that $\Delta x = x_1 - x_0 = 1$. Then the equation of the line is

$$y = y_0 + m(x - x_0) = y_0 + \Delta y_0(x - x_0),$$

where $\Delta y_0 = y_1 - y_0$ is called the first difference. Just as two points determine a line, three points determine a quadratic function, provided that the points are not collinear. The quadratic polynomial that passes through the three points (x_0, y_0), (x_1, y_1), and (x_2, y_2), with $\Delta x = x_1 - x_0 = x_2 - x_1 = 1$, is

$$P_2(x) = y_0 + \Delta y_0(x - x_0) + \frac{\Delta^2 y_0}{2!}(x - x_0)(x - x_1), \quad (2)$$

where $\Delta^2 y_0$ is the second difference,

$$\begin{aligned} \Delta^2 y_0 &= \Delta(\Delta y_0) = \Delta(y_1 - y_0) \\ &= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0. \end{aligned}$$

Equation (2) is known as the quadratic *Newton (forward) interpolating polynomial*. For instance, if we have the three points (1, 12), (2, 10), and (3, 14), then $\Delta y_0 = y_1 - y_0 = 10 - 12 = -2$, and $\Delta^2 y_0 = y_2 - 2y_1 + y_0 = 14 - 2(10) + 12 = 6$. The resulting quadratic is

$$P_2(x) = 12 - 2(x - 1) + 3(x - 1)(x - 2).$$

The graph of this polynomial along with the three interpolating points is shown in **figure 3**.

To see a derivation of Newton's formula, consider the points (x_0, y_0), (x_1, y_1), and (x_2, y_2) holding $\Delta x = 1$. The linear function

$$P_1(x) = y_0 + \Delta y_0(x - x_0)$$

passes through the first two points, (x_0, y_0) and (x_1, y_1), but not the third point, (x_2, y_2).

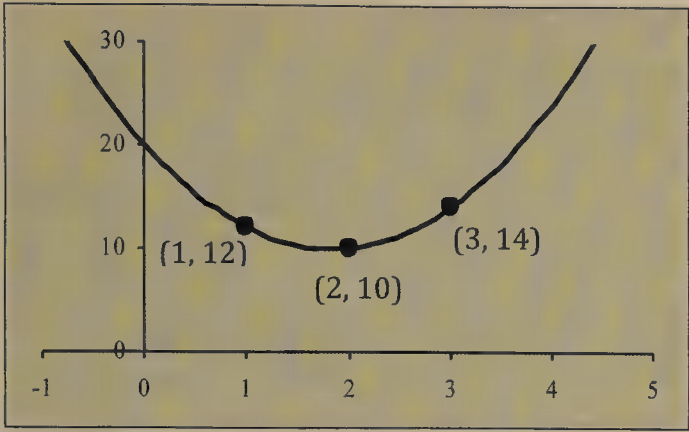


Fig. 3 Uniform increments in x are required to use Newton's interpolation method.

At $x = x_2$,

$$P_1(x_2) = y_0 + \Delta y_0(x_2 - x_0) = 2y_1 - y_0.$$

The point $(x_2, P_1(x_2))$ lies at a vertical distance D units above or below (x_2, y_2), where

$$D = \left|y_2 - P_1(x_2)\right| = \left|y_2 - (2y_1 - y_0)\right| = \left|y_2 - 2y_1 + y_0\right|.$$

The direction above or below depends on whether $y_2 - 2y_1 + y_0$ is less than or greater than zero.

We will modify $P_1(x)$, retaining the fit to the first two points, by adding a quadratic function $g_2(x)$ that has real zeros x_0 and x_1. Moreover, at $x = x_2$, we want the value of $g_2(x)$ to be $y_2 - 2y_1 + y_0$ to close the "gap" of $|y_2 - 2y_1 + y_0|$. Under these conditions, we have

$$P_1(x_0) + g_2(x_0) = y_0 + 0 = y_0,$$

$$P_1(x_1) + g_2(x_1) = y_1 + 0 = y_1,$$

and

$$P_1(x_2) + g_2(x_2) = (2y_1 - y_0) + (y_2 - 2y_1 + y_0) = y_2.$$

Consequently, the new function $P_2(x) = P_1(x) + g_2(x)$ passes through all three points.

At the same time, we know that $g_2(x)$ must be of the form $g_2(x) = k(x - x_0)(x - x_1)$, where k is a constant that can be determined by the condition $g_2(x_2) = y_2 - 2y_1 + y_0$. By formula (1) with $p = x_2$ and $q = y_2 - 2y_1 + y_0$, we have

$$g_2(x) = (y_2 - 2y_1 + y_0)\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{y_2 - 2y_1 + y_0}{2 \cdot 1}(x - x_0)(x - x_1).$$

Recall that $\Delta^2 y_0 = y_2 - 2y_1 + y_0$ and $\Delta x = x_1 - x_0 = x_2 - x_1 = 1$. Therefore,

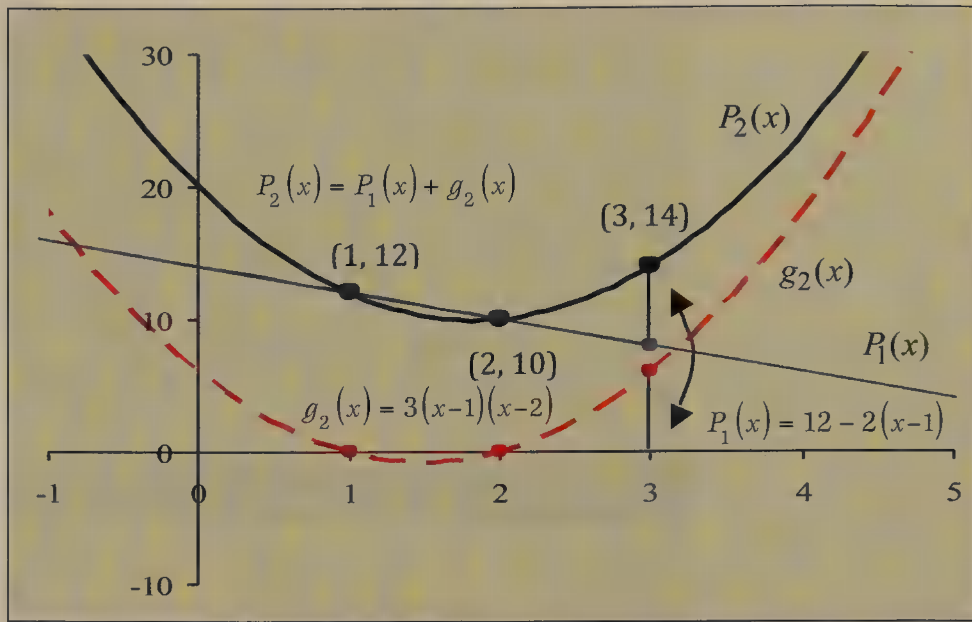


Fig. 4 The quadratic correction $g_2(x)$ has zeros where the linear function $P_1(x)$ attains y-values of the specified points.

$$k = \frac{\Delta^2 y_0}{2!} \text{ and } g_2(x) = \frac{\Delta^2 y_0}{2!}(x - x_0)(x - x_1).$$

Again, with the interpolating points (1, 12), (2, 10), and (3, 14) and the quadratic interpolating polynomial $P_2(x) = 12 - 2(x - 1) + 3(x - 1)(x - 2)$, we show the graph of $P_2(x)$ along with its two component functions $P_1(x)$ and $g_2(x)$ (see **fig. 4**). Note that $P_1(3)$ is 6 units below the desired parabola and that $g_2(3) = 6$.

Similarly, the four points, (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3), again with $\Delta x = 1$, determine the cubic Newton interpolating polynomial

$$P_3(x) = y_0 + \Delta y_0(x - x_0) + \frac{\Delta^2 y_0}{2!}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!}(x - x_0)(x - x_1)(x - x_2).$$

In general, a set of $n + 1$ points (x_0, y_0), (x_1, y_1), . . . (x_n, y_n) with constant spacing of 1 between the x-values determines a polynomial of degree n (unless the points fall on a curve of lower degree), and the equation of that polynomial can be found by Newton's formula. If the points are uniformly spaced with nonzero spacing $\Delta x = h \neq 1$, then a simple extension of Newton's formula applies; for instance, the cubic interpolating polynomial is

$$P_3(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2).$$

One principal characteristic of any linear function is that the successive differences of the y-values are always constant, provided that the x-values are

uniformly spaced. There are some generalizations of this criterion. In particular, if all the successive second differences $\Delta^2 y_i$ ($i = 0, 1, 2, \ldots$) of the y-values are constant, then the points fall into a quadratic pattern; if all the successive third differences $\Delta^3 y_i$ ($i = 0, 1, 2, \ldots$) of the y-values are constant, then the points fall into a cubic pattern; and so forth.

If we are given a set of $n + 1$ points with uniformly spaced x-values, we can determine the degree of an interpolating polynomial by constructing a table of differences. The first higher-degree difference $\Delta^m y_i$ ($i = 0, 1, 2, \ldots$) for which all values are the same (but not zero) shows that the polynomial is of degree m. When $m < n$, the points fall onto a polynomial of lower degree than n.

On the other hand, if the difference table has no column of constants, so that the last possible column of $\Delta^n y_i$ contains only one entry, $\Delta^n y_0$, which is not zero, then all the points lie on a polynomial of degree n but not on any single polynomial of lower degree. More significantly, the entries in the difference table, along with Newton's interpolating formula, immediately give us the equation of that polynomial, as we demonstrate later.

Table 1 Difference Table for a Set of Points

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	–12	–27	–6	12	0
2	–39	–33	6	12	0
3	–72	–27	18	12	0
4	–99	–9	30	12	0
5	–108	21	42	12	
6	–87	63	54		
7	–24	117			
8	93				

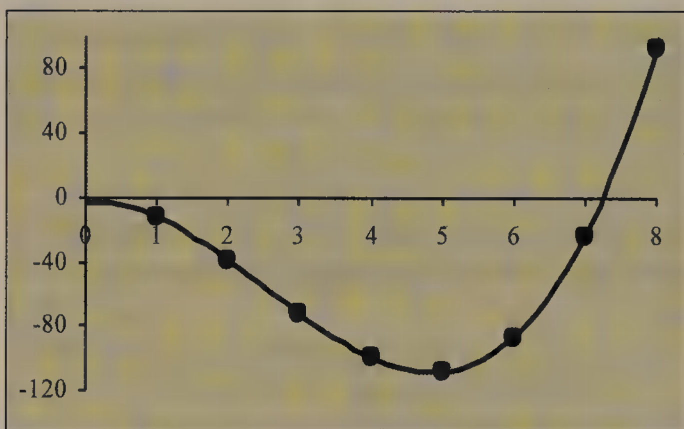


Fig. 5 A cubic interpolating polynomial passes through the points in **table 1**.

Consider the eight points $(x_0, y_0), \ldots (x_7, y_7)$ in **table 1**, where there is a constant difference between all the x-values. We have extended the table to include columns for the first, second, third, and fourth differences. Notice that all the entries under $\Delta^3 y$ are equal to 12; this tells us that the points fall on a cubic polynomial. (All subsequent higher-order differences will then be zero.) Moreover, the coefficients in Newton's third-degree interpolation formula (2) are y_0, Δy_0, $\Delta^2 y_0/2!$ and $\Delta^3 y_0/3!$. These are all based on the first row of the table: $y_0 = -12$, $\Delta y_0 = -27$, $\Delta^2 y_0/2! = -6/2 = -3$, and $\Delta^3 y_0/3! = 12/6 = 2$. Therefore, Newton's cubic interpolating polynomial is

$$P_3(x) = -12 - 27(x-1) - 3(x-1)(x-2) + 2(x-1)(x-2)(x-3).$$

The graph of this cubic, along with the interpolating points, is shown in **figure 5**.

Figure 6 gives a different perspective by showing not only the interpolating polynomial but also the linear function based on the first two points and the quadratic function based on the first three points. Thus, the line through the first two points is the linear Newton polynomial

$$P_1(x) = -12 - 27(x - 1) = -27x + 15,$$

which is precisely the first two terms of $P_3(x)$. The parabolic curve is the quadratic Newton polynomial

$$P_2(x) = -12 - 27(x - 1) - 3(x - 1)(x - 2),$$

which is the first three terms of $P_3(x)$. And the cubic is the full interpolating polynomial. Thus, each additional term in the interpolating polynomial can be thought of as an adjustment to the preceding polynomial that "picks up" one more point.

In addition, the authors have created an interactive Excel® spreadsheet (go to **www.nctm.org/mt060**) that allows teachers and students to investigate the ideas relating data values, tables of differences, and Newton's interpolating polynomials. The spreadsheet allows the user to select the number of data points (up to 12), constructs the table of differences, draws the graph of the interpo-

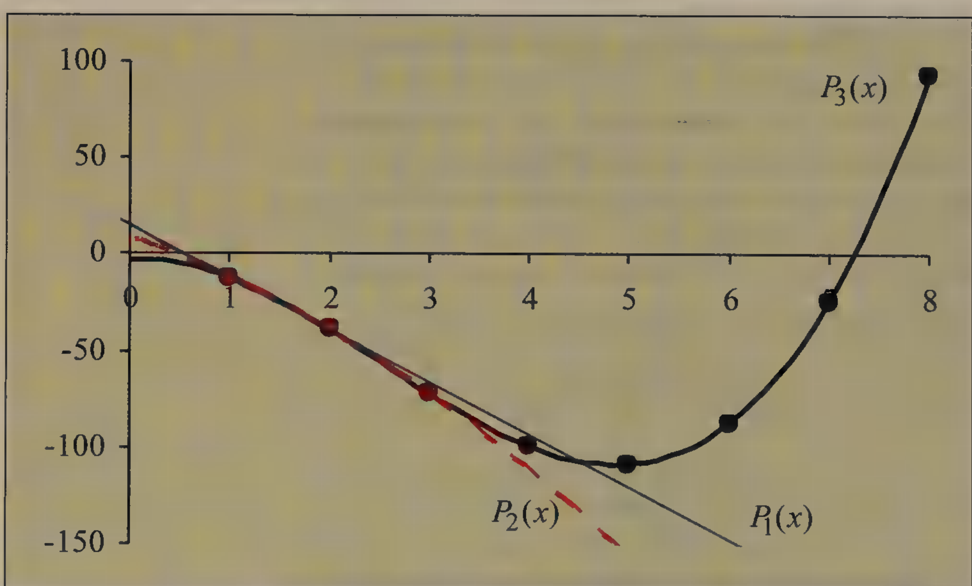


Fig. 6 The graphs of $P_1(x)$, $P_2(x)$, and $P_3(x)$ show that higher-degree terms allow subsequent polynomials to match additional points.

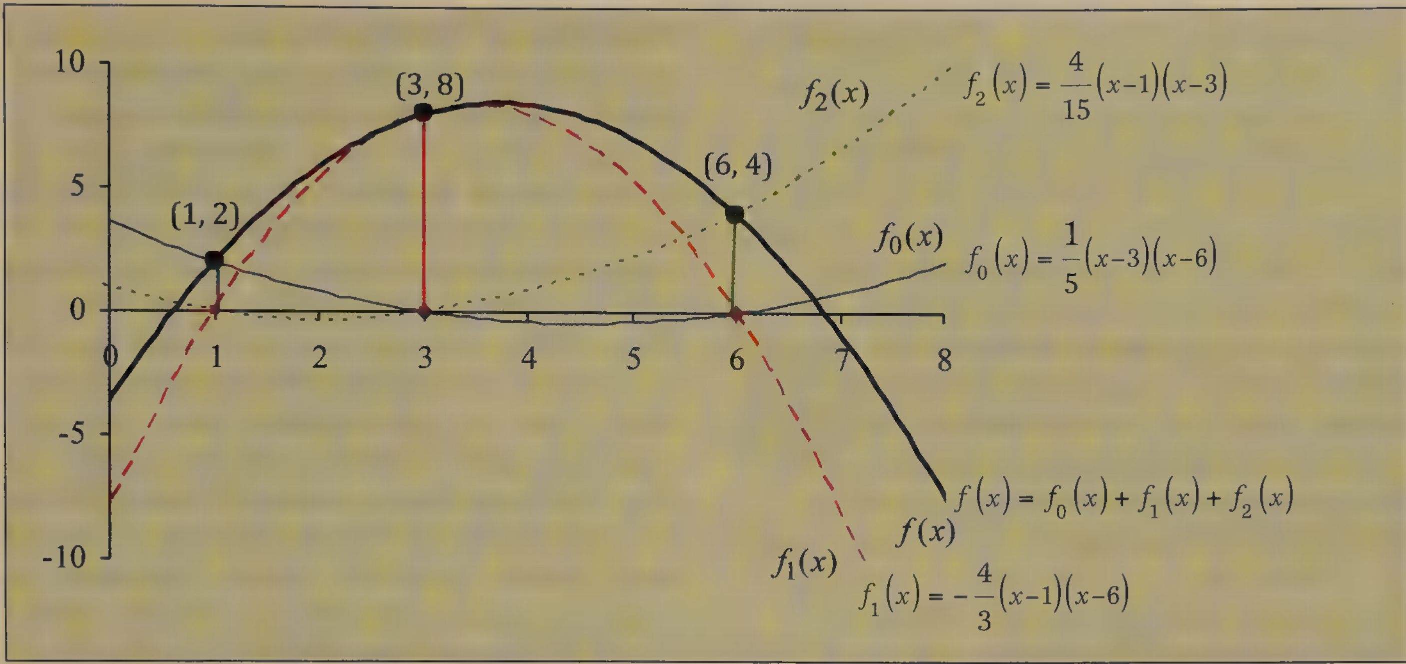


Fig. 7 The three component quadratic functions of $f(x)$ have nonzero values at one of x_0, x_1, or x_2.

lating polynomial, and displays the equation of the polynomial.

LAGRANGE'S INTERPOLATION

The major drawback to Newton's interpolation formula is the fact that it requires uniform spacing for the x-values in a set of data. An alternative approach is Lagrange's interpolation formula, which does not require uniform spacing. But it carries with it a cost—it is a more complicated formula that usually involves considerably more computational effort.

Suppose that we have three points (x_0, y_0), (x_1, y_1), and (x_2, y_2), where all x_i are different. The observation in the earlier discussion on the connection between real zeros and their linear factors suggests that we may decompose the quadratic function $f(x)$ into three quadratic functions—$f_0(x)$, $f_1(x)$, and $f_2(x)$—by looking at the values of the function at x_0, x_1, and x_2 in the following way:

$$\begin{array}{lccccccc} f(x_0) = & y_0 & + & 0 & + & 0 & = & y_0 \\ f(x_1) = & 0 & + & y_1 & + & 0 & = & y_1 \\ f(x_2) = & 0 & + & 0 & + & y_2 & = & y_2 \\ & \downarrow & & \downarrow & & \downarrow & & \end{array}$$
$$f(x) = f_0(x) + f_1(x) + f_2(x)$$

Looking at the display above, we pair each of the three numbers in the first column—y_0, 0, and 0—with x_0, x_1, and x_2, respectively, to obtain the three points (x_0, y_0), $(x_1, 0)$, and $(x_2, 0)$. These points specify the quadratic function

$$f_0(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

by formula (1). Similarly, the function $f_1(x)$ is specified by the three points $(x_0, 0)$, (x_1, y_1), and $(x_2, 0)$, whereas $f_2(x)$ is specified by $(x_0, 0)$, $(x_1, 0)$, and (x_2, y_2). The expressions for these two functions are

$$f_1(x) = y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

and

$$f_2(x) = y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

The sum of $f_0(x)$, $f_1(x)$, and $f_2(x)$ is the desired function that passes through the original three points (x_0, y_0), (x_1, y_1), and (x_2, y_2).

The equation of the parabola that passes through the three points (1, 2), (3, 8), and (6, 4) is therefore

$$\begin{aligned} f(x) &= 2\frac{(x-3)(x-6)}{(1-3)(1-6)} + 8\frac{(x-1)(x-6)}{(3-1)(3-6)} + 4\frac{(x-1)(x-3)}{(6-1)(6-3)} \\ &= \frac{1}{5}(x-3)(x-6) - \frac{4}{3}(x-1)(x-6) + \frac{4}{15}(x-1)(x-3) \\ &= -\frac{13}{15}x^2 + \frac{97}{15}x - \frac{18}{5}. \end{aligned}$$

Figure 7 shows $f(x)$ along with its three component functions: $f_0(x)$, $f_1(x)$, and $f_2(x)$. Let's focus on the three solid dots that represent the given points. First, consider the point (1, 2). Notice that only the graphs of $f(x)$ and $f_0(x)$ pass through this point; at $x = 1$, the other two curves have x-intercepts. Similarly, only the graphs of $f(x)$ and $f_1(x)$ meet at (3, 8); the other two curves have x-intercepts at $x = 3$. Finally, only the graphs of $f(x)$ and $f_2(x)$ intersect at (6, 4); the other two share the x-intercept at $x = 6$.

The formula

$$f(x)=2\frac{(x-3)(x-6)}{(1-3)(1-6)}+8\frac{(x-1)(x-6)}{(3-1)(3-6)}+4\frac{(x-1)(x-3)}{(6-1)(6-3)}$$

is an example of the quadratic *Lagrange interpolating formula* and is named after Joseph-Louis Lagrange, a famous Italian-born French mathematician of the eighteenth century (Atkinson 1989). More generally, the quadratic Lagrange interpolating polynomial that passes through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2) is

$$L_2(x)=y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}+y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}+y_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}.$$

The ideas discussed here can be extended if more than three points are given. For instance, the four points (x_0, y_0), (x_1, y_1), (x_2, y_2), and (x_3, y_3) determine a cubic polynomial (assuming that all x_i are distinct and that the points do not lie on a line or a parabola). An extension of the Lagrange interpolation formula used above gives an immediate way to construct this cubic. The third-degree Lagrange interpolating polynomial, $L_3(x)$, is composed of four cubic components—$f_0(x)$, $f_1(x)$, $f_2(x)$, and $f_3(x)$—each constructed in the comparable way. The result is

$$\begin{aligned}L_3(x)=&\,y_0\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}+y_1\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\\&+y_2\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}+y_3\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}.\end{aligned}$$

Notice that, at each of the four interpolating points, only one of the four cubic components is not zero and so contributes precisely the associated value of y at each of those points. The other three cubic components contribute zero at these points. For instance, at $x = x_0$, only the first component is nonzero, and it contributes y_0 to the sum. That is, $L_3(x_0) = y_0$. It is worth noting that the Lagrange formula works and provides the correct polynomial even if a polynomial of degree less than n fits the data.

The authors also provide an interactive spreadsheet to investigate graphically and numerically the way in which the linear, quadratic, and cubic Lagrange polynomials are constructed from their component functions for user-defined sets of data; go to **www.nctm.org.mt061**.

POLYNOMIAL REGRESSION

Often in practice we have large sets of data. If we have $n + 1$ points (where n is large), the interpolating polynomial is of degree n, presuming that the points do not fall onto a polynomial of lower degree. This high-degree polynomial certainly passes through all the interpolating points, but it can be a very poor match between those points. Such a situation can occur because the polynomial may shoot way up or down after passing through each interpolating point to reach a turning point before coming back to hit the next interpolating point. The interpolating polynomial may change direction up to $n - 1$ times. We illustrate such a case in **figure 8**, where the interpolating points are (0, 5), (1, 9), (2, –10), (3, 40), (4, 5), (5, 50), and (6, 29). Such an oscillatory behavior may dramatically affect the accuracy of approximation between interpolating points and make the function very sensitive to any changes of the interpolating points.

If the exact fit to the specified points is not a condition, we may circumvent these difficulties by finding a lower-degree polynomial that will give reasonable accuracy. Polynomial regression is a common approach to capture the overall trend in a data set and potentially give much better approximations *between* the interpolating points, even though the regression polynomial does not necessarily pass through any of them.

PEDAGOGICAL CONSIDERATIONS

Polynomials have been extremely important in mathematical applications for centuries. A common use is in approximation, and this role has increased dramatically in our technological age. For instance, many data sets (such as the spread of diseases including AIDS) follow polynomial patterns; all calculators and computational programs use polynomial approximations to find the values of transcendental functions; most numerical integration techniques are based on polynomial approximations that in turn involve interpolation methods; all

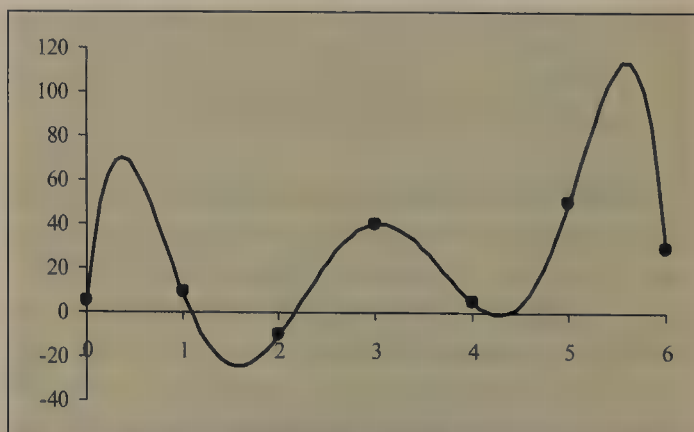


Fig. 8 Oscillatory behavior of interpolating polynomials may exaggerate extreme values, as shown for x-values between 5 and 6.

computer graphics programs (including Paint) use cubic splines (the notion of creating apparently smooth curves that pass through a series of points by finding and graphing a series of cubic curves that connect smoothly to one another at their endpoints). Many financial companies use cubic polynomials to calculate trends in data. Students should see some of these ideas to demonstrate such a pervasive, long-lasting, and important application of mathematics.

At the same time, the approaches used here provide the opportunity to reinforce some major ideas and methods in algebra and precalculus mathematics. Through our discussion, we touch on many important concepts at the introductory level of mathematics—for instance, the connection between the real zeros of a polynomial and both its linear factors and the x-intercepts of the graph of the polynomial; combining functions (more than just the mechanics); and systems of linear equations. This is a wonderful opportunity for students to see seemingly unrelated mathematics topics used to solve a routine problem such as fitting a quadratic function to three points. Perhaps equally important, it gives us a chance to introduce students to one of the most widely used applications of polynomials, in contrast to the usual spectrum of algebraic techniques for manipulating and especially factoring all manner of arcane expressions that almost never arise outside algebra classes.

CURVE-FITTING FORMULAS IN THE PRECALCULUS CURRICULUM

Within mathematics, the Newton and Lagrange interpolating formulas provide a foundation for the development of methods in numerical integration and differentiation, approximation theory, and the numerical solution of differential equations. Consequently, they become very important in the interpolation theory of numerical analysis. However, these ideas can also be valuable additions to a modern course in algebra or precalculus.

Moreover, curve fitting has become an important topic in modern algebra and precalculus classes, although it is usually approached almost exclusively from the perspective of regression. And polynomial regression is usually limited by the available technology—graphing calculators up to fourth degree and Excel up to sixth degree.

Some readers have undoubtedly projected possible connections between ideas here and topics in calculus. We investigate those parallels in the appendix.

The Newton and Lagrange formulas can be discussed through a sequence of "What if?" questions. The precalculus classroom is a place to foster deep learning of mathematics by using different topics together to investigate rich problems like polynomial curve fitting.

REFERENCES

Atkinson, Kendall E. 1989. *An Introduction to Numerical Analysis.* 2nd ed. New York: Wiley and Sons.

Boole, George. 1860, 2007. *A Treatise on the Calculus of Finite Differences.* New York: Cosimo Classics.

Mickens, Ronald. 1991. *Difference Equations: Theory and Applications.* 2nd ed. Chapman and Hall/CRC.

YAJUN YANG, yangy@farmingdale.edu, is a professor of mathematics at Farmingdale State College in Farmingdale, New York. She is interested in making precalculus and calculus more meaningful to students and in incorporating technology into the classroom. **SHELDON P. GORDON,** gordonsp@farmingdale.edu, is SUNY Distinguished Teaching Professor at Farmingdale State University. He has been deeply involved in efforts to renew the mathematics curriculum at all levels and is the author of about a dozen books and almost two hundred articles.

APPENDIX: SOME CONNECTIONS TO CALCULUS

There are many connections between ideas in this article and major concepts in calculus, and readers are likely to have thought of some of these possibilities. The long history of a discrete analogue to the continuous theory of calculus and differential equations dates back to the days of Isaac Newton. Where the familiar approach to calculus is based on smooth functions defined on a continuum—either a finite interval or the real line—the parallel theory of finite differences and difference equations is based on sequences (often sets of data). Almost all the major ideas from the continuous case have comparable interpretations in the discrete case. We do not dwell on this here, but we suggest that interested readers consult such texts as the classic treatise by George Boole (best known for Boolean algebra) from 1860 or the more modern text by Mickens (1991).

First, in discussing difference tables and how they relate to the degree of a polynomial, we pointed out that if a set of points with uniform increment in x falls into a polynomial pattern of degree n, then the nth differences of the y-values will be constant. The $n + 1$st differences and all subsequent differences will be zero. This is the discrete analogue of the comparable fact in calculus that the nth derivative of a polynomial of degree n is a constant and all subsequent derivatives will be identically zero.

Second, Newton's interpolation formula of degree 3—say,

$$P_3(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2)$$

with $\Delta x = h$—is obviously very similar to the formula for the third-degree Taylor polynomial for a smooth function $y = f(x)$ near $x = x_0$:

$$T_3(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$

Let's see just how close the two are.

Consider what happens to the Newton interpolating formula $P_3(x)$ as $h \to 0$. The expressions $\Delta^k y_0/h^k$ in the polynomial converge toward $f^{(k)}(x_0)$, the successive derivatives of the function at $x = x_0$, in the limit. Moreover, as $h \to 0$, all the x_i approach $x = x_0$, although they do retain the uniform spacing. As all the points x_i coalesce at $x = x_0$, we see that the various factors $(x - x_i)$ in Newton's formula all converge toward $(x - x_0)$, and so the various products approach the successive powers of $(x - x_0)$. Thus, the Taylor polynomial expansion of a function is the limit of the Newton interpolating polynomials as $h \to 0$. The comparable argument applies to the Newton polynomial of any degree n.

Developing a comparable argument for Lagrange's interpolation formula is not quite so simple. However, if we consider the quadratic Lagrange interpolating polynomial that passes through the points with uniform spacing (x_0, y_0), $(x_1, y_1) = (x_0 + h, y_1)$, and $(x_2, y_2) = (x_0 + 2h, y_2)$, we find the following:

$$\begin{aligned}
L_2(x) &= y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\
&\quad + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \\
&= y_0 \frac{(x - x_0 - h)(x - x_0 - 2h)}{(-h)(-2h)} + y_1 \frac{(x - x_0)(x - x_0 - 2h)}{(h)(-h)} \\
&\quad + y_2 \frac{(x - x_0)(x - x_0 - h)}{(2h)(h)} \\
&= y_0 \frac{(x - x_0)^2 - h(x - x_0) - 2h(x - x_0) + 2h^2}{2h^2} \\
&\quad + y_1 \frac{(x - x_0)^2 - 2h(x - x_0)}{-h^2} + y_2 \frac{(x - x_0)^2 - h(x - x_0)}{2h^2} \\
&= \frac{y_0 - 2y_1 + y_2}{2h^2}(x - x_0)^2 + \frac{-3y_0 + 4y_1 - y_2}{2h}(x - x_0) + y_0 \\
&= \frac{\Delta^2 y_0}{2h^2}(x - x_0)^2 + \frac{-3y_0 + 4y_1 - y_2}{2h}(x - x_0) + y_0
\end{aligned}$$

Let's look at $L_2(x)$ as $h \to 0$. We already know that $\Delta^2 y_0/h^2 \to f''(x_0)$. The expression in the second term is

$$\begin{aligned}
\frac{-3y_0 + 4y_1 - y_2}{2h} &= \frac{-3y_0 + 3y_1}{2h} + \frac{y_1 - y_2}{2h} \\
&= \frac{3}{2}\frac{y_1 - y_0}{h} - \frac{1}{2}\frac{y_2 - y_1}{h},
\end{aligned}$$

where

$$\frac{y_1 - y_0}{h} \to f'(x_0) \text{ and } \frac{y_2 - y_1}{h} \to f'(x_1) \text{ as } h \to 0.$$

Since we asserted that f was smooth, then $f''(x_0)$ exists and $f'(x)$ is continuous near $x = x_0$. As $h \to 0$ (that is, as $x_1 \to x_0$), we have $f'(x_1) \to f'(x_0)$. Therefore, the expression becomes

$$\frac{3}{2}f'(x_0) - \frac{1}{2}f'(x_0) = f'(x_0).$$

Consequently, in the limit as $h \to 0$, we find that the limit of the quadratic Lagrange interpolation polynomial is the quadratic Taylor polynomial.

We presume that the comparable argument can be extended for higher-order polynomials. However, considering the complexity of this argument in the case where the interpolating points are uniformly spaced, we doubt that there will be an easy argument to demonstrate the comparable result where the interpolating points are not uniformly spaced.

From a somewhat different perspective, we should expect that, if the interpolation points are equally spaced, then Lagrange's interpolating polynomial would reduce to Newton's formula. Let's consider the case with quadratics, so that the Lagrange formula reduces to

$$L_2(x) = \frac{\Delta^2 y_0}{2h^2}(x - x_0)^2 + \frac{-3y_0 + 4y_1 - y_2}{2h}(x - x_0) + y_0,$$

and the Newton formula is

$$P_2(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1).$$

Notice that the coefficients of x^2 will be the same and that the constant terms are the same. To compare the linear coefficients, notice that

$$\begin{aligned}
-3y_0 + 4y_1 - y_2 &= 2y_1 - 2y_0 - y_2 + 2y_1 - y_0 \\
&= 2(y_1 - y_0) - (y_2 - 2y_1 + y_0) \\
&= 2\Delta y_0 - \Delta^2 y_0,
\end{aligned}$$

so that

$$L_2(x) = \frac{\Delta^2 y_0}{2h^2}(x - x_0)^2 + \frac{2\Delta y_0 - \Delta^2 y_0}{2h}(x - x_0) + y_0.$$

Further, we can rewrite the expression for $P_2(x)$ as powers of $(x - x_0)$:

$$\begin{aligned}
P_2(x) &= y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) \\
&= y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}\left[(x - x_0)(x - x_0 + x_0 - x_1)\right] \\
&= y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}\left[(x - x_0)^2 - h(x - x_0)\right] \\
&= y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)^2 - \frac{\Delta^2 y_0}{2!h}(x - x_0) \\
&= y_0 + \left[\frac{\Delta y_0}{h} - \frac{\Delta^2 y_0}{2!h}\right](x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)^2
\end{aligned}$$

We therefore see that both representations reduce to the same expression. However, the comparable argument with $n = 3$ or more, will be increasingly complicated.

more4U

For an Excel spreadsheet using macros to show the Newton polynomial, download one of the free apps for your smartphone and then scan this tag to access **www.nctm.org/mt060**. For an Excel spreadsheet showing the Lagrange polynomial, scan this tag to access **www.nctm.org/mt061**.

Katherine A. Linsenmeier, Miriam Sherin, Janet Walkoe, and Martha Mulligan

Lenses for Examining Students' Mathematical Thinking

A few years ago, a colleague shared a video from a first-year algebra class that he had observed. The video captured a class discussion about slopes of horizontal and vertical lines. At the beginning of the discussion, the teacher, Ms. Milner, asks, "How can we have a slope of zero?" Students respond in various ways; one student, Peter, explains that a horizontal line would have a slope of zero "because it's never moving up." Later, Milner asks about the slope of a vertical line, and another student, Alex, replies that because "it went . . . up and down and didn't move at all, it would be zero." Milner then asks the class about the slope of a line through the points (0, 0) and (0, 5). Peter says that "the slope is zero, because you subtract the change in x and the change in y. On the top there'd be zero and on the bottom there'd be 5, and any division problem that has zero in it has to be zero." Finally, Rafael disagrees and suggests that the slope is undefined because "in division there can't be a number over zero."

We were intrigued. What do these students understand about slope? What does Peter mean by "it's never moving up," and what does Alex mean when he says "it . . . didn't move at all"? To us, simply saying that Peter is initially correct, that Rafael is correct at the end, and that Alex is wrong does not capture the complexity of these students' thinking. Further, focusing on the surface-level correctness of their answers, rather than exploring the nuances of what the students are thinking, would not provide Milner with the information she needs to advance her students' understanding.

Of course, understanding students' thinking can be challenging, particularly in the moment of instruction (Sherin, Jacobs, and Philipp 2011). Students' ideas can be complex, and students may not always articulate their ideas clearly. The need to move the lesson forward and to attend to so many aspects of the classroom at once can leave a teacher with little time to think in depth about a student's comment. In addition, although the phrase "making sense of student thinking" is commonly used in teacher education and professional development, often little guidance is offered for doing so (Ball and Cohen 1999).

Connecting Research to Teaching appears in alternate issues of *Mathematics Teacher* and brings research insights and findings to the journal's readers. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more information on the department and guidelines for submitting a manuscript, please visit http://www.nctm.org/publications/content.aspx?id=10440#connecting.

Department editors
Laurie Cavey, *lauriecavey@boisestate.edu, Boise State University, Boise, ID; and* ***Michael Weiss,*** *mweiss@msu.edu, Michigan State University, East Lansing, MI*

For more than a decade, we have developed and studied video clubs as a context for helping mathematics teachers learn to attend more closely to the ideas that students raise in the classroom. In a video club, teachers meet to watch and discuss excerpts of video from their own classes (Sherin 2000). Our work with video clubs is driven by the belief that video can serve as a valuable resource for teachers, providing a window into classroom interactions within a time frame that allows for reflection. In addition, watching video excerpts with colleagues can provide opportunities to consider multiple perspectives on students' ideas.

We have studied video clubs with elementary school teachers (van Es 2009), middle school teachers (Sherin and Han 2004), and high school mathematics teachers (Walkoe 2013) and with both preservice and in-service teachers. Most teachers whom we have worked with have been involved in a single cycle of video club meetings over a school year or university term, although some have participated in video clubs for more than one year.

Our early work documented that, throughout a series of video club meetings, teachers' discussions shifted away from focusing primarily on issues of pedagogy and management to paying more attention to students' thinking as expressed in the video (Sherin and Han 2004). Further, initially teachers described (e.g., "she said . . .") and evaluated (e.g., "he was wrong") the thinking that they noticed; however, in the later video club meetings, teachers primarily interpreted (e.g., "he probably meant . . .") students' ideas. These findings were based on qualitative analyses of transcripts from video club meetings.

More recently, in collaboration with our colleague Elizabeth van Es, we investigated the effect of video club participation on teachers' classroom practices. We found that the increased focus on student thinking that teachers develop in the video club setting does influence their instruction. Teachers are able to bring an emphasis on better understanding student thinking back to their classrooms; in particular, they are more open to soliciting multiple student responses and more likely to follow up on those responses (van Es and Sherin 2010). These findings were the result of examining videotapes of whole-class discussions in participants' classrooms before and after the video club meetings. We coded these classroom discussions for the extent to which teachers attended to students' mathematical thinking as well as for the ways in which they did so during instruction.

In our most recent work, reported here, we wanted to better understand how video clubs support this kind of learning among teachers. To do so, we videotaped a series of video club meetings with secondary school mathematics teachers and then conducted qualitative analyses of those meetings. In particular, we highlighted and then categorized the different approaches that the facilitators and teachers used to make sense of the student ideas that were discussed. As a result of this analysis, we identified three lenses that teachers used repeatedly to make sense of student ideas expressed in the video clubs. Specifically, teachers frequently began by *taking an initial look* at a student's idea before *going deeper* and finally by *looking across* a single student's solutions to several different problems or across several students' solutions to a single problem. In our ongoing research, we are making these lenses explicit to video club facilitators, who report that they are a productive tool for supporting teachers in learning to interpret student thinking (Sherin et al. 2013).

Martha: Let's see if we can decipher what Alex is saying.
Allison: Well, he said, "It didn't move at all," so I'm wondering if he thinks whether horizontally or vertically, . . . if it doesn't move, that means zero . . . if one of the two directions doesn't move, then it's zero.
Martha: So he's not making the distinction between *x*s moving or changing and *y*s moving or changing?
Allison: He didn't say which one. He just said, "It went up and down and didn't move at all." So I'm assuming he meant it didn't move left and right. So since it didn't move in some direction, the slope was zero.
Martha: Now where do you think he would get that idea from?
Sheila: He may not have had a name for an undefined slope. I feel like he maybe is also mixing up horizontal and vertical, because he says, "because it's horizontal," and then he talks about the line going up and down.
Rick: I think it actually kind of ties together with Peter, that it doesn't matter where the zero is. If you're not moving in one direction, the slope is zero. Kind of like how Peter says . . . if you have a slope of zero in the division problem, it's zero automatically.

Fig. 1 Teachers in a video club meeting discuss students' thinking.

THE THREE LENSES EXPLAINED

To better understand the three lenses, we will reconsider the slope discussion from Milner's class shared earlier but this time from the perspective of a video club discussion. This brief class discussion provides a number of student ideas about slope that teachers can productively explore in a video club meeting (Linsenmeier and Sherin 2009). **Figure 1** contains an excerpt from a video club discussion about Milner's class. Martha is the facilitator for this meeting; the other participants are teachers. In this excerpt, teachers are talking about the ideas shared by Alex and Peter.

Taking an Initial Look

An important first step in exploring a student's idea is to try to interpret the meaning of what the student is saying or doing (Cavey and Mahavier 2010; Lampert 2001). In particular, taking an initial look involves exploring the specifics of what a student may or may not understand. For example, when teachers discuss Peter's explanation—"because it's never moving up"—they can explore what he means when he says that the line is not moving. Is he confused about the slope formula or about division involving zero? In investigating these questions, the teachers move beyond simply concluding that Peter does not know the slope of a vertical line to trying to understand what he does know. Keep in mind that it can be valuable to

Taking an Initial Look	What is the student's idea? ▪ What different comments does Peter make in the video? ▪ What does he mean by "It's never moving up"?
	What misunderstanding does the student exhibit? ▪ Does Peter believe that any division problem with a zero in it, anywhere, is equal to zero?
	What is reasonable about this idea? ▪ What does Peter understand about division problems involving zero? • Is there a way to interpret Peter's comment about division with zero so that it makes sense?

Fig. 2 Taking an initial look at a student's thinking raises specific questions.

Going Deeper	Where did this idea come from? ▪ Where do you think Peter would get that idea?
	Is there more than one way to understand what the student is saying? ▪ How else can we interpret what Peter meant? ▪ Do you think that Peter believes that a zero slope and an undefined slope are the same?

Fig. 3 Going deeper into a student's thinking prompts interpretation of the student's reasoning.

identify correct thinking in the middle of an idea that appears incorrect (Sherin 2002). **Figure 2** suggests some specific initial questions that teachers can ask about Peter's thinking in this video clip.

Alex's comments about slope also prove interesting to examine. At the beginning of the excerpt shown in **figure 1**, Martha prompts the group to "see if we can decipher what Alex is saying." Allison focuses on Alex's statement that "it didn't move at all" and wonders if not moving "means zero" to Alex. Notice that, to make sense of Alex's ideas, the teachers begin to hypothesize about Alex's reasoning. They are trying to determine what understanding he is using to justify his claims, what misunderstanding he might have about mathematics, and what about his reasoning is sensible. That is, they are taking an initial look.

Going Deeper

Once teachers have determined what a student seems to be saying and seems to understand, they might pursue the potential origins of the student's idea or whether there are alternative hypotheses about the student's reasoning. This is what we mean by going deeper.

Figuring out what might have prompted a student to think in a particular way can clarify the reasoning and is often an important step in trying to help a student reformulate his or her own thinking (Cohen 2004). When teachers consider the source of Peter's idea about the slope of a vertical line, they might ask whether he is building on Alex's comment or whether he is focusing on calculating slope using the two points provided by Milner (see **fig. 3**). As seen in the video club discussion of Alex's thinking (see **fig. 1**), the teachers go deeper when they respond to Martha's question, "Where do you think he would get that idea from?" Sheila offers a few alternatives, suggesting that perhaps Alex did not have "a name for undefined slope" or that he might have been "mixing up horizontal and vertical."

In addition, considering alternative interpretations of a student's idea can be useful for teachers (van Es 2009). When we are in the middle of instruction, we have a tendency to assume that we understand what a student is saying. Are we sure that our initial conclusions are correct, or might there be something else going on? Even when it is not possible to reach a definitive answer, practicing this skill of stepping back and questioning our own assumptions can help teachers be more open and more aware of the range of ways that students are thinking (Sherin and van Es 2009). **Figure 3** offers some questions that prompt teachers to think about multiple ways to interpret Peter's comments in the video clip.

Looking Across

As we work to interpret a specific student idea, it can be helpful to make connections to other ideas raised in class, either ideas expressed by other students in the class or other ideas expressed by the same student (Sherin and Han 2004). Trying to identify connections across multiple expressed ideas is what we mean by looking across.

In particular, exploring how ideas expressed by different students are related can provide additional insight into a particular student's comment. For example, both Peter and Alex talk about lines "not moving." Teachers might wonder whether the two students necessarily mean the same thing. In addition, do we get a more nuanced view of what Peter might have meant when we examine Alex's thinking? In the video club meeting, Rick looks across when he suggests that Alex's idea "kind of ties together with Peter." Rick explains that for both these students, "not moving in one direction" means that "the slope is zero" (see **fig. 1**).

In addition to looking across different students' ideas, considering how an individual student is thinking about multiple ideas can be productive. For instance, teachers can look at what Peter says about horizontal lines and compare his comment with what he says about vertical lines. In addition, teachers might ponder how a student would respond to new questions not posed in the video. For example, would Peter say that a horizontal line or a vertical line is steeper? Such questions can help teachers analyze what a student understands as well as prepare teachers to ask effective, probing questions during instruction.

See **figure 4** for some questions designed to support teachers as they examine Peter's thinking across contexts.

Looking Across	How does this idea compare with others? • Do Peter and Alex have the same idea about slope? • How does Rafael's understanding of slope compare with Peter's or Alex's?
	How might this student respond to another problem? • How would Peter talk about the slope of a line that is not vertical or horizontal? • How would Peter decide which of two given lines has a bigger slope?

Fig. 4 Looking at the student's thinking across contexts helps teachers analyze the student's understanding and develop effective questioning.

PUTTING THE LENSES TO USE

In presenting these three lenses, we have tried to take the abstract idea of "making sense of student thinking" and turn it into a more manageable and concrete task. Our work with teachers in video clubs shows that productive discussions arise when teachers ask questions across these three lenses. Moving forward, we are making the lenses explicit to facilitators and teachers to help guide teacher discussion of students' mathematical thinking.

Notice that in **figure 1** the video club participants use the lenses sequentially, starting with an initial look ("Let's see if we can decipher what Alex is saying"), then going deeper ("Now where do you think he would get that idea from?"), and finally looking across ("I think it actually kind of ties together with Peter"). We have also seen teachers have an extended discussion using a single lens or engage in several shorter cycles of an initial look, going deeper, and looking across during a single video club meeting.

Discussions of student thinking may also proceed in a less linear fashion, moving, for example, from an initial look directly to looking across or even from looking across back to going deeper or an initial look. This might happen when a desire to compare two students' thinking prompts us to look more closely at an individual student's idea. For example, we might not consider Alex's understanding of slope until we begin to wonder whether Peter is building on what Alex says.

We have found that using the lenses in different sequences does not constrain the depth of the discussions that take place. Instead, what is most important in supporting productive discussion of student thinking is for teachers to have opportunities to engage in all three strategies over time (Sherin 2007).

Also notice that in the discussion excerpt in **figure 1**, the facilitator, Martha, prompts the use of both taking

an initial look and going deeper, but it is a participant, Rick, who prompts the group to look across. We have found this shift to be a common one. In a new video club, it is most often the facilitator who prompts participants to apply the different lenses. Over time, however, teachers take on the three lenses themselves and require less specific guidance from a facilitator (Walkoe 2013).

TOOLS FOR UNDERSTANDING STUDENT THINKING

The three lenses presented here can be valuable tools for supporting teachers as they strive to make sense of student thinking. Although we have used these lenses primarily in the context of video clubs, they are equally applicable in other professional development settings that involve examining student work. We have found that teachers who use these strategies in a professional development setting often transfer this practice to their everyday teaching by applying the lenses in their classrooms. (Sherin and van Es 2009). As Laura reflected, "I wonder how many times I just hear 'zero' and I'm like, 'Yup, good job.' [Now] I wonder what they actually said . . . I need to [think about] this more."

Finally, although engaging in such detailed reasoning on your own is possible, teachers seem to value the opportunity to engage with peers as they use these lenses. As Sheila explained, "Having people to talk it over with, I feel like I'm getting more out of it than just [thinking about] a set of questions . . . on my own." Rick also emphasized that having the opportunity to hear different interpretations was important to him: "There were a couple of times where [my colleagues] interpreted it one way, and . . . I didn't even think of it that way, but [then I thought] 'Maybe that is what the kid meant.'"

Teachers interested in being part of a video club can start by inviting a few colleagues to watch and discuss a video from their classroom and then seeing whether those same colleagues would be willing to share a video of their own. Look for classroom episodes for which you think it would be productive to have your colleagues help you explore student ideas—that is, video clips in which student thinking is visible. Keeping the three lenses in mind can then support you and your colleagues as you begin to reflect on student thinking in a deep and meaningful way.

REFERENCES

Ball, Deborah L., and David K. Cohen. 1999. "Developing Practice, Developing Practitioners: Toward a Practice-Based Theory of Professional Education." In *Teaching as the Learning Profession*, edited by Linda Darling-Hammond and Gary Sykes, pp. 3–31. San Francisco: Jossey-Bass.

Cohen, Sophia. 2004. *Teachers' Professional Development and the Elementary Mathematics Classroom: Bringing Understandings to Light*. Mahwah, NJ: Lawrence Erlbaum.

Cavey, Laurie O., and William T. Mahavier. 2010. "Seeing the Potential in Students' Questions." *Mathematics Teacher* 104 (2): 133–37.

Lampert, Magdelene. 2001. *Teaching Problems and the Problems of Teaching*. New Haven and London: Yale University Press.

Linsenmeier, Katherine A., and Miriam G. Sherin. 2009. "What Makes a Video Clip Interesting?" *Teaching Children Mathematics* 15 (7): 418–22.

Sherin, Miriam G. 2000. "Viewing Teaching on Videotape." *Educational Leadership* 57: 36–38.

———. 2002. "A Balancing Act: Developing a Discourse Community in a Mathematics Classroom." *Journal of Mathematics Teacher Education* 5: 205–33.

———. 2007. "The Development of Teachers' Professional Vision in Video Clubs." In *Video Research in the Learning Sciences*, edited by Ricki Goldman, Roy Pea, Brigid Barron, and Sharon Derry, pp. 383–95. Hillsdale, NJ: Lawrence Erlbaum.

Sherin, Miriam G., Elizabeth Dyer, Janet D. K. Walkoe, and Rosemary S. Russ. 2013. "Using Video Clubs to Examine Student Thinking about Algebra." Presentation at the Annual Meeting of the National Council of Teachers of Mathematics, Denver, Colorado, April.

Sherin, Miriam G., and Sandra Han. 2004. "Teacher Learning in the Context of a Video Club." *Teaching and Teacher Education* 20: 163–83.

Sherin, Miriam G., Victoria R. Jacobs, and Randolph A Philipp, eds. 2011. *Mathematics Teacher Noticing: Seeing through Teachers' Eyes*. New York: Routledge.

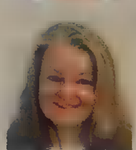
Sherin, Miriam G., and Elizabeth A. van Es. 2009. "Effects of Video Club Participation on Teachers' Professional Vision." *Journal of Teacher Education* 60 (1): 20–37.

van Es, Elizabeth A. 2009. "Participants' Roles in the Context of a Video Club." *Journal of the Learning Sciences* 18 (1): 100–137.

van Es, Elizabeth A., and Miriam G. Sherin. 2010. "The Influence of Video Clubs on Teachers' Thinking and Practice." *Journal of Mathematics Teacher Education* 13: 155–76.

Walkoe, Janet D. K. 2013. "Investigating Teacher Noticing of Student Algebraic Thinking." Ph.D. diss., Northwestern University. http://search.proquest.com/docview/1328169315?accountid=131239 (1328169315)



KATHERINE A. LINSENMEIER, k-linsenmeier@northwestern.edu, teaches mathematics at New Trier High School in Winnetka, Illinois, and studies mathematics teaching and learning at Northwestern University in Evanston, Illinois. **MIRIAM SHERIN,** msherin@northwestern.edu, is a professor of learning sciences in the School of Education and Social Policy at Northwestern University. Her interests include mathematics teaching and learning, teacher cognition, and the use of video for teacher learning. **JANET WALKOE,** jwalkoe@umd.edu, is an assistant professor of mathematics education at the University of Maryland in College Park. Her research interests include teacher noticing and teacher responsiveness in the mathematics classroom. **MARTHA MULLIGAN,** MLMulligan@ncps.edu, is a teacher and the chair of the mathematics department at Northside College Preparatory High School in Chicago. She served as teacher-in-residence at Northwestern University during the 2009-10 school year.

NCTM Proudly Presents

MATH IS ALL AROUND US

Principles to Actions: Ensuring Mathematical Success for All

What will it take to turn the opportunity of the Common Core State Standards for Mathematics into reality in every classroom, school, and district?

Continuing its tradition of mathematics education leadership, NCTM has undertaken a major initiative to define and describe the principles and actions, including specific teaching practices, that are essential for a high-quality mathematics education for all students.

This landmark new title offers guidance to teachers, mathematics coaches, administrators, parents, and policymakers:

- Provides a research-based description of eight essential Mathematics Teaching Practices
- Describes the conditions, structures, and policies that must support the Teaching Practices
- Builds on NCTM's *Principles and Standards for School Mathematics* and supports implementation of the Common Core State Standards for Mathematics to attain much higher levels of mathematics achievement for all students
- Identifies obstacles, unproductive and productive beliefs, and key actions that must be understood, acknowledged, and addressed by all stakeholders
- Encourages teachers of mathematics to engage students in mathematical thinking, reasoning, and sense making to significantly strengthen teaching and learning

www.nctm.org/PrinciplestoActions

© April 2014, Stock #14861
List Price: $28.95 | **Member Price: $23.16**

SAVE 25%! $21.71

**Use code MT814 when placing order.
Offer expires 10/31/14.***

Also available as an eBook
List Price: $4.99 | **Member Price: $3.99**

INSIDE

Progress and Challenge
Effective Teaching and Learning
Essential Elements
- Access and Equity
- Curriculum
- Tools and Technology
- Assessment
- Professionalism

Taking Action
References

Go to link at left to access full Table of Contents, Preface, and an Excerpt.

*This offer reflects an additional 5% savings off list price, in addition to your regular 20% member discount.

NCTM Members Save 25%! Use code MT814 when placing order. Offer expires 10/31/14.*

NCTM | NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Visit www.nctm.org/catalog for tables of content and sample pages.

For more information or to place an order,
please call **(800) 235-7566** or visit **www.nctm.org/catalog**.

TECHNOLOGY tips

Ashley Nahornick

"You're the Author" Textbooks

Textbooks and e-books are the mainstay of many classrooms, and often textbooks dictate the scope of content and style of instruction in the classroom (Tyson and Woodward 1989). Although the current trend in education is paperless books, we need to understand how students learn from textbooks. Tyson and Woodward (1989) point out that "perhaps the most damning findings are those concerning the failure of textbooks to expose students adequately to the process of scientific inquiry" (p. 15). Discussions of the cost, ease of use, equity, and portability of textbooks are typical (Baumann 2010). Nevertheless, we need to be open to different ways of instruction to ensure students' learning.

One new approach to the traditional textbook is the innovative textbook *Interactive Science* (Buckley et al. 2012). *Interactive Science* is a combination of the traditional textbook, an e-book, and a workbook and includes many write-in sections in which the student becomes the author. The textbook has two parts: (1) a hardcopy book with numerous sections, examples, and questions to be filled in by the student; and (2) an online textbook with special features such as a digital laboratory for experiments. *Interactive Science* is an excellent example of a textbook that encourages students to learn in different ways and could easily be brought into any classroom. Students feel a sense of ownership of their learning because they are the authors. This is a unique learning experience that we as educators should consider emulating.

WHAT CAN WE LEARN FROM "YOU'RE THE AUTHOR" TEXTBOOKS?

Learning is more than copying down notes and regurgitating information; it should be an active process. Students already write comments in the margins of their textbooks, so why not encourage them to write down their thoughts, solve problems, and learn in their textbooks? Write-in textbooks, or "you're the author" textbooks, keep students engaged and lessons short. Salman Khan, the founder of the ever-popular Khan Academy, considers long lectures ineffective and recommends a change in classroom activities every fifteen minutes or so (Khan 2012). Having the students "write" the textbook ensures that they will be engaged.

Students can answer questions, draw diagrams, or reflect on their learning, all in their write-in textbook. This all-in-one model helps students organize their work, create connections, and keep things simple. Students will not need extra note pads or paper, and the chances of losing that important piece of paper are reduced.

Becoming authors also gives students a strong sense of empowerment. They take pride in their work. Darryl Yong, a mathematician at Harvey Mudd College, chronicled and published his experience of teaching in a large urban high school during his sabbatical for the American Mathematics Society.

Technology Tips, which provides a forum for innovative uses of technology in the teaching and learning of mathematics, appears seven times each year in *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submission, visit http://www.nctm.org/publications/content.aspx?id=10440#tech.

Department editors
James Kett, *j.gkett@gmail.com, Singapore-American High School (retired), Singapore; and* ***Andrew Benzing****, abenzing@wssd.org, Strath Haven High School, Wallingford, PA*

Yong (2012) argues that student self-concept is the best predictor of student success. Student empowerment helps promote further learning, build confidence, and use resources effectively. Svitak (2012) offers suggestions for improving student self-concept including conducting class forums, giving students choices, using technology, and having debates. Many of these suggestions can be included in "you're the author" textbooks.

"You're the author" textbooks are a transition between traditional textbooks and contemporary learning initiatives. Traditional aspects remain by including a physical book; however, the write-in aspect allows for flexibility, innovation, and creativity by the student. Robinson (2012), a prominent educationalist, insists in his TED talk (discussed in his article) that creativity in schools is paramount but not always fostered. Write-in textbooks give students the opportunity to search for gaps in knowledge, look for missing elements in a problem, search for solutions, make conjectures, and formulate hypotheses.

HOW CAN I HAVE THESE BENEFITS IN MY CLASSROOM?

There are a variety of ways to bring "you're the author" methodology into the classroom. A relatively easy and straightforward solution would be to purchase or rent textbooks similar to *Interactive Science* for your class. However, if new textbooks are not in the budget, you could create your own write-in textbooks for your class by combining worksheets, problem sheets, and activities that focus on the students being the author. The activities should be more than fill-in-the-blank. The problems should allow for original thought, not just be a repetition of an algorithm. The write-in textbook could act as both notebook and textbook for students. It could be a collection of handouts or online documents. Creating your own write-in textbooks for students may be time-consuming but may be more rewarding because they would fit perfectly with your students' needs.

HOW CAN I USE TECHNOLOGY WITH THIS?

Technology can be easily leveraged to create a "you're the author" opportunity for students. For example, using Google Documents (more commonly known as Google Docs™), teachers can create notes, files, or text for the students to annotate. You can create documents, presentations, spreadsheets, forms, or drawings. Google Docs is a subset of Google Drive™ and is free as long as you sign up for a free Google account. To sign up for a Google account, if you do not already have one, go to the link https://accounts.google.com/SignUp). Once you have a Google account, you can select **Drive** from the square grid apps icon on the top-right corner of the screen (see **fig. 1**).

Once Google Drive is open, you will see any documents that you have loaded to the drive as well as a red **Create** icon in the top-left portion of the screen. Selecting the **Create** tab displays a menu showing the various Google applications available, including but not limited to

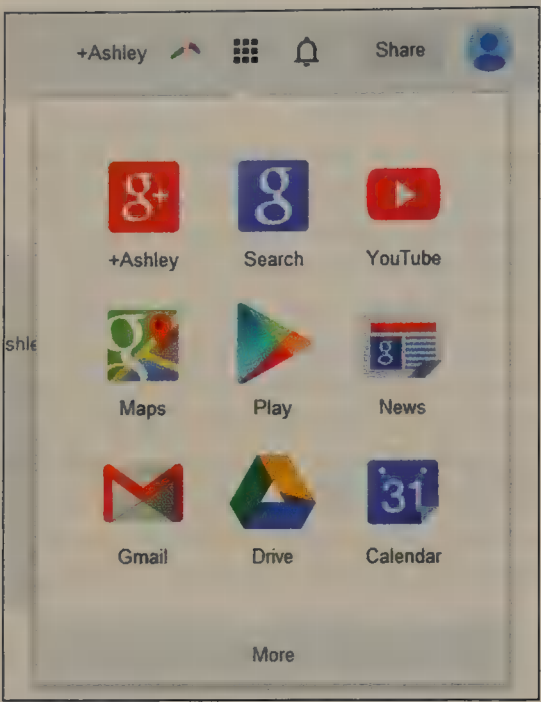


Fig. 1 The apps icon is a square grid that expands to allow users to select and open Google Drive.

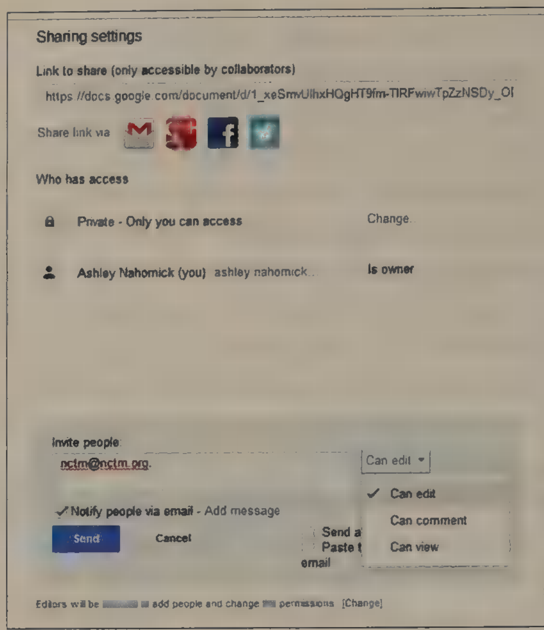


Fig. 2 **Share** settings allow users to manage access to their documents.

Documents, Presentations, Spreadsheets, and Forms. Google invites you to connect your Drive account to a growing number of other Google Drive–friendly apps as well. By selecting Documents, you are redirected to the word processing app and are working in a Google Document. This document is linked to your Drive account and is saved in real time. Here, many of the features that you have grown accustomed to from other robust word processors are included in an intuitive way, resulting in relative ease of use.

You will also notice a **Share** icon in the top-right portion of the document as well. Selecting this icon offers the author the opportunity to manage the accessibility of the document, ranging from an open public status to allowing only those invited, through link or e-mail, to view or change the document. Here is where you could include e-mails addresses of students, colleagues, or parents who would be interested in viewing or annotating the document.

Google Docs allows multiple users to edit, publish, and share their work simultaneously for free. Another beneficial feature of Google Docs is that the students' work will never be lost; documents are saved automatically. Students would be able to write their thoughts and ideas directly into the document and easily share it with their teacher or classmates. To share a document, just click on the **Share** icon in the top-right corner and select your sharing setting, such as the people you would like to share the document with and their editing privileges (see **fig. 2**).

In the mathematics classroom, student reflection is very important. Many students believe that mathematics is rigid and unquestionable. Steele (2007), who conducted a teaching experiment in which students wrote about the problem-solving process, suggests having students write about their mathematics problem-solving experience to have a better understanding. As a part of "You're the Author" worksheets or textbooks, students would explain what they are doing, give reasons for what they are doing, and show what they are thinking. This would be an excellent way to encourage student reflection.

Howland, Jonassen, and Mara (2012) write extensively on meaningful learning with technology and maintain that it is important for technologies to be leveraged as a vehicle for exploring new knowledge and to support learning by constructing. Using technology such as Google Docs in the context of "you're the author" experiences can provide opportunities for students to explore patterns, formulate conjectures, and interact with unfamiliar situations. Some iPad® apps, such as Evernote®, could also be used in place of or in conjunction with Google Docs.

"You're the author" textbooks are an innovative way to transition between traditional and contemporary learning environments. By giving students greater ownership of their learning, we give them a way to improve their critical thinking and understanding.

REFERENCES

Baumann, Michael. 2010. "Ebooks: A New School of Thought." *Information Today* 27 (5): 1–4.

Howland, Jane, David H. Jonassen, and Rose M. Mara. 2012. *Meaningful Learning with Technology*. 4th ed. Boston: Pearson.

Buckley, Don, Zipporah Miller, Mike Padilla, Kathryn Thornton, Grant Wiggins, and Michael Wysession. *Interactive Science*. 2012. Upper Saddle River, NJ: Pearson Prentice Hall.

Khan, Salman. 2012. "Why Long Lectures Are Ineffective." *Time*. http://ideas.time.com/2012/10/02/why-lectures-are-ineffective/

Robinson, Ken. 2012. "Do Schools Kill Creativity?" *Huffington Post*. http://www.huffingtonpost.com/sir-ken-robinson/do-schools-kill-creativity_b_2252942.html

Steele, Diana F. 2007. "Understanding Students' Problem-Solving Knowledge through Their Writing." *Mathematics Teaching in the Middle School* 13 (2): 102–9.

Svitak, Adora. 2012. "Five Ways to Empower Students." *Edutopia*. http://www.edutopia.org/blog/empower-students-adora-svitak

Tyson, Harriet, and Arthur Woodward. 1989. "Why Students Aren't Learning Very Much from Textbooks." *Educational Leadership* 47 (3): 14–17.

Yong, Darryl. 2012. "Adventures in Teaching: A Professor Goes to High School to Learn about Teaching Math." *Notices of the American Mathematical Society* 59 (10): 1408–15. doi:http://dx.doi.org/10.1090/noti906

ASHLEY NAHORNICK, ashley.nahornick@gmail.com, recently completed her doctorate in education with a specialization in mathematics education at Teachers College, Columbia University, in New York City. Her passion is mathematical problem solving and improving instructional practice in mathematics classrooms.

2016 Focus Issue

Teaching Mathematics Online

Today's students live in a world of computers, tablets, smartphones, and cloud computing. The pace of innovation continues to accelerate. How can mathematics teachers capitalize on these trends for the good of our students? The Editorial Panel of *Mathematics Teacher* invites teachers, teacher educators, education researchers, and others to share their experiences in online mathematics education. Submissions that highlight how online technologies can support NCTM's Equity Principle are especially encouraged. In some cases, print alone will not adequately capture the total impact of the article; in such cases, having active links or other dynamic resources will be optimal.

All possible innovative uses of online resources cannot be enumerated here, but the following questions will get the conversation started.

OPPORTUNITY

How can online technologies—

- be leveraged for mathematics learning?
- provide for student collaboration across distances?
- improve professional development opportunities?
- affect the achievement gaps between different groups of students?

IMPLEMENTATION

How do you—

- remain connected to your students when teaching online?
- use specific online tools to help students learn?
- balance and complement traditional classroom formats with online technologies?
- compare the effectiveness of different methods for student learning?

These are just a few of our ideas; we want to know what you think. We particularly encourage teachers to share their own favorite online lessons.

Please submit manuscripts and Web materials at **mt.msubmit.net** by **May 1, 2015.** Be sure to enter the call's title (Teaching Mathematics Online) in the Department/Calls field. No author identification should appear in the text of the manuscript. Manuscript guidelines are available at www.nctm.org/publications/content.aspx?id=22602. If you have ideas related to this topic and wish to discuss them before sending a manuscript or need details about restrictions on online resources as part of submissions, please contact Tara Slesar (tslesar@nctm.org).

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

CALL FOR MANUSCRIPTS

DELVING deeper

Robert Zawada

Turning Points

Traditionally in calculus, the maximum number of turning points in the graph of a one-variable polynomial function is presented as $(n - 1)$, where n is the degree of the polynomial. However, by using Descartes' rule of signs and certain properties of polynomial functions, we can establish a better estimate than the $(n - 1)$ rule provides.

Our estimate of the maximum number of turning points could be stated as follows:

$$T_{max} = 2S + C + Z, \text{ where}$$

- S denotes the number of occurrences where the degrees of two successive terms in the polynomial have the same parity but the coefficients have different signs. (One could think of S as "same powers/sign change.")
- C marks the number of occurrences where degrees of two successive terms in the polynomial change parity, excluding constant term. (One could think of C as "change in powers.")
- Z is an indicator variable that we set equal to 1 in the absence of a linear term. If the polynomial does contain a linear term, we set $Z = 0$.

With the same notation, we can also deduce a formula for the maximum possible number of inflection points:

$$P_{max} = 2S + C + Z - 1 = T_{max} - 1$$

Delving Deeper offers a forum for classroom teachers to share the mathematics from their own work with the journal's readership; it appears in every issue of *Mathematics Teacher*. Manuscripts for the department should be submitted via http://mt.msubmit.net. For more background information on the department and guidelines for submitting a manuscript, visit http://www.nctm.org/publications/content.aspx?id=10440#delving.

Department editors

Maria Fung, *mfung@worcester.edu, Worcester State College, MA;* ***Ted Hodgson,*** *hodgsont1@nku.edu, Northern Kentucky University, Highland Heights, KY; and* ***Brian M. Dean,*** *bdean@pasco.k12.fl.us, District School Board of Pasco County, FL*

EXAMPLE 1

Consider the polynomial $f(x) = x^6 + 3x^4 - 7x^3 - 5x^2 + 6x + 5$, whose graph is given in **figure** 1.

The $(n - 1)$ and $(n - 2)$ rules state that this new polynomial function, too, has a maximum of 5 turning points and 4 inflection points. The formula, however, produces a maximum of 3 turning points and 2 inflection points: $T_{max} = 2S + C + Z = 2 \cdot 0 + 3 + 0 = 3$ and $P_{max} = T_{max} - 1 = 2$.

The number of turning points in a graph is related to both the complexity and the oscillatory behavior of the function. For centuries, mathematicians have sought to increase the accuracy of solutions to polynomial equations as well as to precisely determine the exact nature of the polynomial functions and their graphs. The factor theorem, the intermediate value theorem, and Descartes' rule of signs are all important results that focus on the roots of polynomial equations.

DESCARTES' RULE OF SIGNS

We can use Descartes' rule to take our knowledge of the maximum number of turning points further. The maximum number of turning points in the graph of a one-variable polynomial function of degree n is typically specified as $n - 1$. This rule

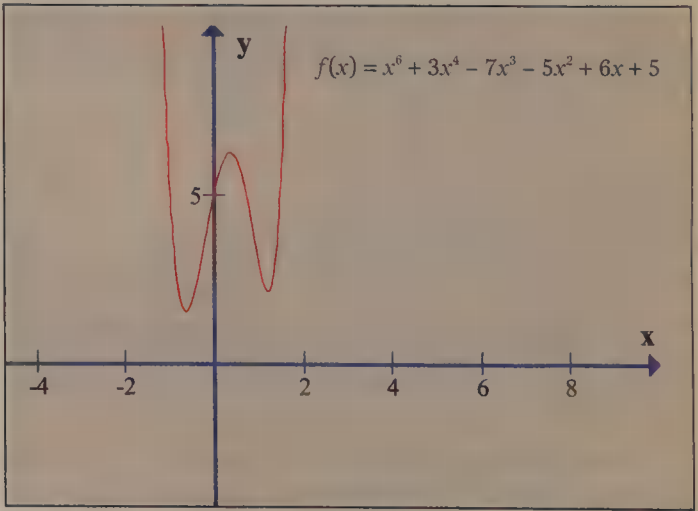


Fig. 1 The $(n - 1)$ rule overestimates the number of turning points for this sixth-degree polynomial.

Table 1 Signs of $f(x)$ and $f(-x)$ in Pairs of Successive Polynomial Terms

Degrees of successive terms have different parity	. . . when coefficients have the same signs	. . . when coefficients have different signs
Odd and even powers	$x^7 + x^6$ (+, +) (–, +)	$x^7 - x^4$ (+, –) (–, –)
Even and odd powers	$4x^4 + 2x^3$ (+, +) (+, –)	$-x^5 + x^2$ (–, +) (+, +)
	Maximum of one negative root	*Maximum of one positive root*
	At most, one root when switching from even-to-odd or odd-to-even powers (rule 1)	
Degrees of successive terms have the same parity	. . . when coefficients have the same signs	. . . when coefficients have different signs
Both terms are even degree	$x^4 + x^2$ (+, +) (+, +)	$x^4 - x^2$ (+, –) (+, –)
Both terms are odd degree	$x^7 + 4x^5$ (+, +) (–, –)	$-x^7 + 4x^5$ (–, +) (+, –)
	No sign changes in the pairs imply no positive or negative roots (rule 3)	*Maximum number of positive roots is 1; maximum number of negative roots is 1; therefore, at most 2 roots (rule 2)*

follows from the fact that the first derivative has degree $n - 1$. However, at times, this rule provides a weak upper bound, especially when the polynomial includes coefficients of zero for many of the powers of x. For example, under the $(n - 1)$ rule, $p(x) = x^{20}$, could have as many as 19 turning points, but in reality there is only 1 turning point, at $x = 0$.

To address this issue with a formula that yields a more precise bound, we need to examine pairs of successive nonzero terms in a polynomial function and the parity of their powers.

We use Descartes' rule of signs:

To find the maximum number of positive roots, consider the polynomial $f(x)$ and add up the number of sign changes.

To find the maximum number of negative roots, consider the polynomial $f(-x)$ and add up the number of sign changes.

We consider three situations:

1. The degrees of successive terms have different parity (alternating from even to odd or odd to even).
2. The degrees of successive terms have the same parity (both even or both odd), but the coefficients have different signs (one positive, one negative).
3. The degrees of successive terms have the same parity (both even or both odd), and the coefficients have the same sign.

When we compare $f(x)$ to $f(-x)$, the sign of the coefficient in a term where the variable is raised to an even power is unaffected. The coefficient of an odd power term will always switch signs when we compare $f(x)$ to $f(-x)$.

In **table 1**, examples of polynomials are given to illustrate these situations. The two sets of ordered pairs of + and – symbols indicate the signs of the coefficients in $f(x)$ and $f(-x)$.

ZAWADA'S FIVE RULES

Now we will use the following rules to obtain the maximum number of possible turning points in a polynomial function—that is, the maximum number of possible roots in the derivative of the polynomial. It follows from the linearity of the derivative and the power rule $d(a_n x^n)/dx = na_n x^{n-1}$ that the derivative will change all the odd powers of the polynomial to even and all the even powers to odd.

Thus, the degrees of terms in the derivative of a polynomial function will alternate in parity the same number of times as in the original polynomial.

The derivative will also have the same number of pairs of successive terms with the same degree parity where a sign change occurs, both in $f(x)$ and in $f(-x)$. Therefore, we can apply Zawada's rules 1 and 2 directly to the original polynomial function. However, interpreting the results as roots of the derivative function, we find that the "maximum number" obtained gives the maximum number of possible turning points.

Rule 1: Any change in the parity of powers between two sequential terms results in at most one root, regardless of the sign of each of the two terms.

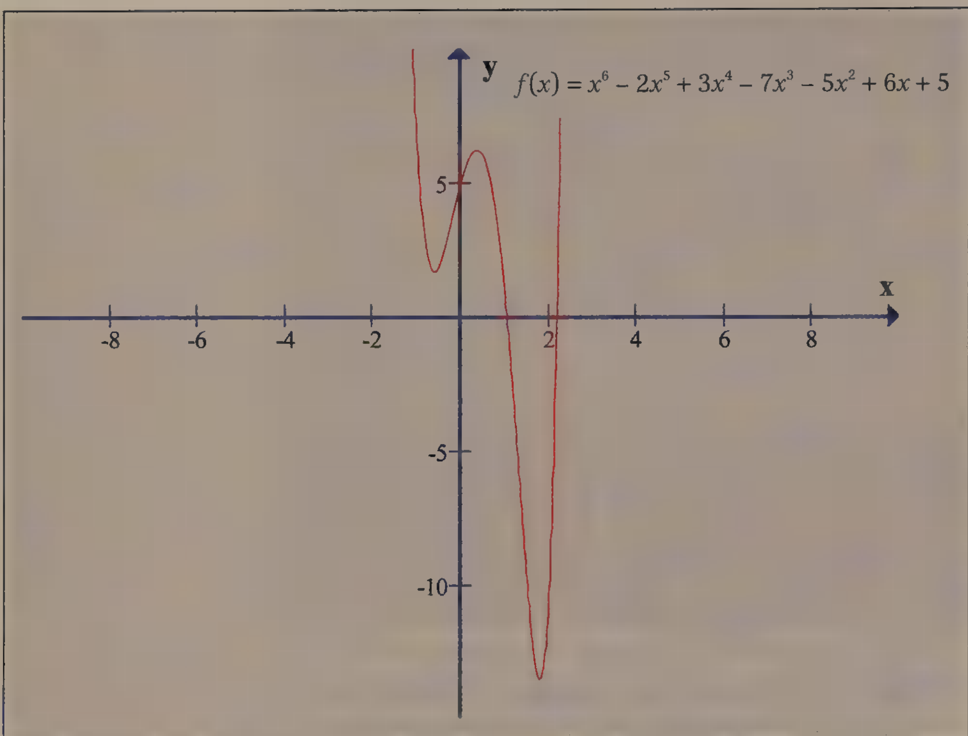


Fig. 2 Both the usual and proposed methods give the same (weak) bound on the number of turning points.

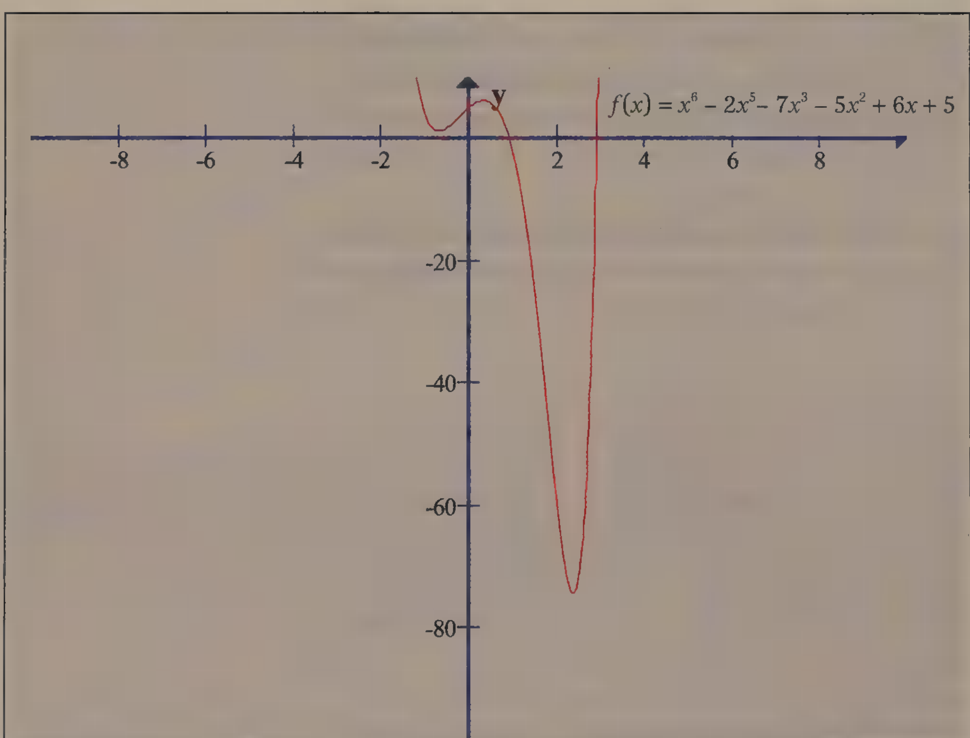


Fig. 3 Because of the "missing" term, the new method provides a better estimate of the number of turning points.

Rule 2: When the powers of two successive terms have the same parity and a change in sign occurs from one term to the next, there will always be at most two roots.

Rule 3: Ignore two successive terms with degrees of the same parity when no sign change occurs between them. This situation will always yield no nonzero roots.

Rule 4: Ignore the constant term in the polynomial. A constant term represents a vertical shift and does not affect the number of turning points or inflection points.

Rule 5: Finally, let Z be an indicator of the absence of a linear term in the polynomial. If the polynomial has no x^1 term, there is no constant in the derivative, so there is an additional root of the derivative—namely, $x = 0$—which may introduce an additional turning point of the polynomial. Thus, Z becomes 1 to represent the additional possibility. If there is an x^1 term, set Z as 0; the presence of a constant in the derivative means that the function cannot have a turning point at $x = 0$.

Reiterating the proposed formula: Let T_{max} denote the maximum number of possible turning points in the graph of a polynomial function. Then, $T_{max} = 2S + C + Z$.

In addition, the number of inflection points in the graph of a polynomial function is at most $(n - 2)$. Because the number of inflection points in a graph is always exactly 1 less than the number of turning points, the maximum number of inflection points may be found more accurately by subtracting 1 from the expression above. Let P_{max} denote the maximum possible number of inflection points. Then, $P_{max} = 2S + C + Z - 1 = T_{max} - 1$.

EXAMPLE 2

Consider the polynomial $f(x) = x^6 - 2x^5 + 3x^4 - 7x^3 - 5x^2 + 6x + 5$, whose graph is given in **figure 2**. The additional x^5 term distinguishes this function from the previous polynomial.

According to the $(n - 1)$ and $(n - 2)$ rules, this sixth-degree polynomial has a maximum of 5 turning points and 4 inflection points.

Observe that we have the following:

- $Z = 0$ because there is a linear term, $6x$.
- $S = 0$ because there are no two successive terms with the same parity of degree.
- $C = 5$ because of the five paired terms: $(x^6 - 2x^5)$, $(-2x^5 + 3x^4)$, $(3x^4 - 7x^3)$, $(-7x^3 - 5x^2)$, and $(-5x^2 + 6x)$.

According to our formulas, then, $T_{max} = 2S + C + Z = 2 \cdot 0 + 5 + 0 = 5$ and $P_{max} = T_{max} - 1 = 4$. The resulting estimates of 5 turning points and 4 inflection points agree with the previous results.

EXAMPLE 3

Now consider a slightly different polynomial whose graph is given in **figure 3**. Here we have dropped the quartic (degree 4) term from the function in **figure 2**.

According to the $(n - 1)$ and $(n - 2)$ rules, this new sixth-degree polynomial still has a maximum of 5 turning points and 4 inflection points. The degree of the first term did not change.

However, the formula accounts for an "internal" change in the polynomial, predicting a maximum of 3 turning points and 2 inflection points: $T_{max} = 2S + C + Z = 2 \cdot 0 + 3 + 0 = 3$ and $P_{max} = T_{max} - 1 = 2$.

EXAMPLE 4

Next, consider the two-term polynomial given in **figure 4**. Here, where most of the lower-degree terms have been eliminated, the usual estimates still suggest a maximum of 5 turning points and 4 inflection points. However, our formula provides better estimates of 2 turning points and 1 inflection point: $T_{max} = 2S + C + Z = 2 \cdot 0 + 1 + 1 = 2$ and $P_{max} = T_{max} - 1 = 1$.

EXAMPLE 5

Finally, consider the eighth-degree polynomial whose graph is given in **figure 5**. The usual estimates suggest a maximum of 7 turning points and 6 inflection points, but our formula returns 5 turning points and 4 inflection points: $T_{max} = 2S + C + Z = 2 \cdot 2 + 0 + 1 = 5$ and $P_{max} = T_{max} - 1 = 4$.

IMPROVED ESTIMATES

We effectively established a series of rules that determine the maximum number of possible turning points of a polynomial function based on the maximum number of possible roots in its derivative. These rules were then condensed into a formula that can be applied directly to a polynomial function to determine the maximum possible number of turning points.

Further examination of this topic could lead to similar findings of the turning points and inflection points in graphs of multivariable polynomial functions.

BIBLIOGRAPHY

Aufmann, Richard N., Vernon C. Barker, and Richard Nation. 1990. *College Algebra*. Boston: Houghton Mifflin.

Stewart, James. 2006. *Calculus: Concepts and Contexts*. 3rd ed. Belmont, CA: Thomson Brooks/Cole.

Eugene, David. 1954. *The Geometry of René Descartes*. New York: Dover.

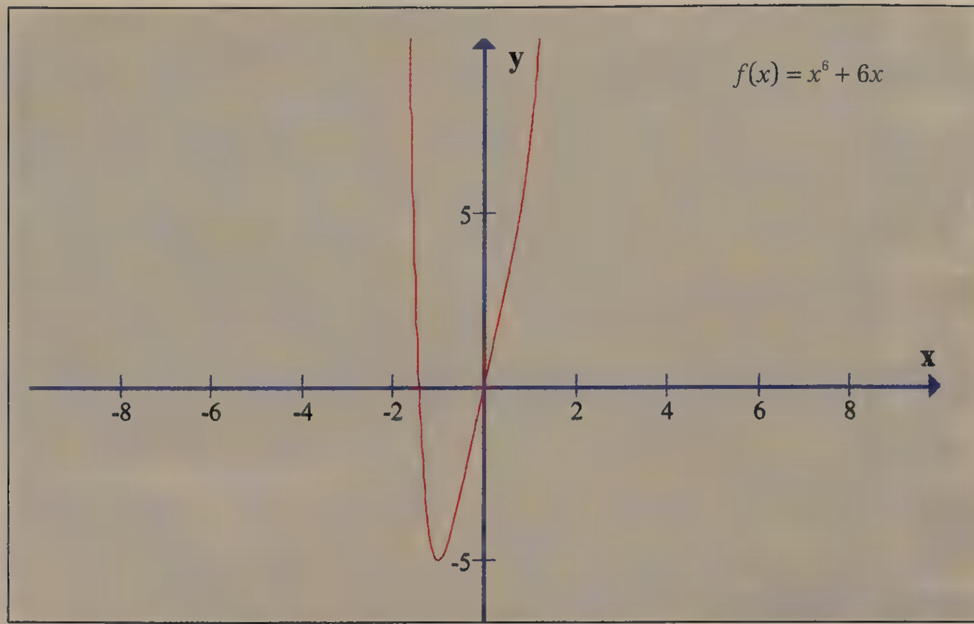


Fig. 4 The $n - 1$ rule is not a good estimate for the number of turning points of this function.

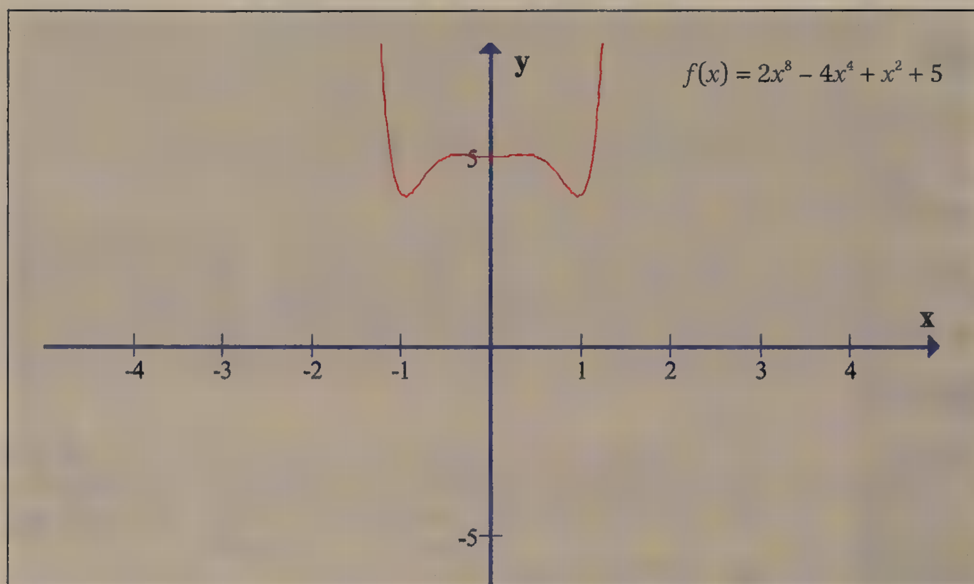


Fig. 5 The new method still overestimates the number of turning points (5) but comes closer to the actual value than the usual estimate (7).

ROBERT ZAWADA, a graduate of Middlesex School in Concord, Massachusetts, is pursuing an undergraduate business degree at the Kenan-Flagler Business School at the University of North Carolina-Chapel Hill.

CALL FOR MANUSCRIPTS

ON THE FRONT BURNER

Emerging Issues in Mathematics Education

This is an exciting time to be a mathematics educator. The recent global scrutiny of mathematics education, forthcoming curricular changes, and evolving technological innovations generate controversial issues. *Mathematics Teacher* is eager to provide a dynamic forum to address topics such as these:

- Assessing students and teachers on the basis of standardized instruments
- Preparing novice and experienced teachers to understand, implement, and evaluate the Common Core State Standards for Mathematics
- Analyzing and interpreting results of international mathematical assessments
- Using novel pedagogical strategies—such as "flipping" a classroom—or using mobile technologies

This list of topics is not all-inclusive. *Mathematics Teacher* welcomes manuscripts that "stir the pot" on provocative current issues in mathematics education.

You may submit your completed manuscript for review by accessing **mt.msubmit.net.** Indicate that the manuscript is being submitted in response to the call On the Front Burner. Be sure to enter the call's title in the Department/Calls field. No author identification should appear in the text of the manuscript. For additional guidelines, go to **www.nctm.org/publications/content.aspx?id=10452.**

NCTM NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

MATHEMATICS teacher

FOR YOUR information

PUBLICATIONS

From NCTM
Individual NCTM members receive a 20 percent discount on NCTM publications. To order, visit the NCTM online catalog at www.nctm.org/catalog or call toll free at (800) 235-7566 to order or request a free print catalog.

Defining Mathematics Education: Presidential Yearbook Selections 1926–2012, Francis (Skip) Fennell and William Speer, eds., 2013. 419 pp., $59.95 cloth. Stock no. 14551. ISBN 978-0-87353-738-4. National Council of Teachers of Mathematics; www.nctm.org.

Six years after the founding of the National Council of Teachers of Mathematics (NCTM) in 1920, an annual yearbook reflecting on topics of mathematics educators' current or anticipated interests was first published. *Defining Mathematics Education* consists of nineteen articles from previous yearbooks, selections recommended by individual past NCTM presidents.

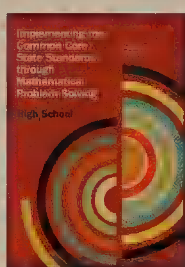
The book serves as a time line of the evolution of mathematics education for these eighty-seven years. Reading through the chapters reveals similarities between past and current issues. For instance, accountability and the progress of algebra and geometry curricula are discussed in the inaugural 1926 yearbook. Adjusting teacher instruction to meet the needs of students of all ethnicities and backgrounds is an underlying theme of many of these articles. Much of the material in this book is applicable to many modern classrooms.

Defining Mathematics Education allows current mathematics educators to understand and appreciate the efforts of our predecessors in ensuring the development of the profession and the mathematics achievement of all students. Further, readers cannot help but reflect on their own teaching and the changes that have occurred over the years.

This book is a wonderful reference and a nostalgic possession for all mathematics educators.

—Mary Robertson
Edison State College
Fort Myers, FL

Implementing the Common Core State Standards through Mathematical Problem Solving: High School, Theresa J. Gurl, Alice F. Artzt, and Alan Sultan, 2012. Foreword by Frances R. Curcio, series ed. 90 pp., $24.95 paper. ISBN 978-0-87353-710-0. Stock no. 14329. National Council of Teachers of Mathematics; www.nctm.org.



How do teachers implement problem solving in their classrooms on a daily basis? This excellent resource can help them do so.

This book provides guidance in teaching both the content standards and the mathematical practices through problem-solving techniques. Two approaches toward problem solving are introduced through tasks. The first is deep problems that give students a starting point in a lesson; the second is a "series of expressions and questions" for discovery of mathematical concepts.

The chapters are organized by topics: algebra, functions, geometry, statistics and probability, and number sense. In each chapter, the authors do an excellent job of introducing the task and then discussing opportunities for teachers to adjust or scaffold the content. This technique gives teachers actual tasks to use in their classrooms as well as strategies that they can fit to other curriculum being taught in the school year. Finally, the appendix does a masterful job of vertically aligning standards from grades six through high school in an easy-to-follow format relating to topic.

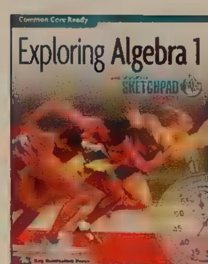
The book also addresses the mathematical practices after every few tasks. This approach can become a bit repetitive; however, it does help solidify the use of the practices in everyday lessons.

I would strongly recommend this easy-to-read guide to any secondary school mathematics teacher. As a resource or a book to study, it would be great for preservice teachers as well as experienced secondary school teachers.

—Carrie Hair
Swope Gifted and Talented Magnet School
Reno, NV

FROM OTHER PUBLISHERS

Exploring Algebra 1 with The Geometer's Sketchpad, Version 5; Exploring Algebra 2 with The Geometer's Sketchpad, Version 5. 2012. $39.95 each, paper. *Exploring Algebra 1*: ISBN 978-1-60440-221-6; *Exploring Algebra 2*: ISBN 978-1-60440-223-0. Key Curriculum Press; www.keycurriculum.com.



These two books provide activities to develop and strengthen algebraic concepts visually through investigations using The Geometer's Sketchpad®. *Exploring Algebra 1* contains activities suited for prealgebra and first-year algebra classes; *Exploring Algebra 2* is suited for first- and second-year algebra and precalculus classes. Each topic within each book contains activity notes, a student activity with answers, and directions for use as a whole-class presentation.

Prices of software, books, and materials are subject to change. Consult the suppliers for the current prices. The comments reflect the reviewers' opinions and do not imply endorsement by the National Council of Teachers of Mathematics.

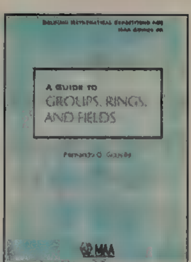
The activities develop concepts by stepping through manipulations and asking questions to help students draw conclusions. Some conclusions are a little harder to extrapolate in the activity and may benefit from a class presentation instead of student exploration. Teachers should review the activity beforehand to determine which option works best for a particular class.

Ready-to-use sketches, available for many activities, allow teachers to spend less time on preparation and more time on exploring relationships. Teachers need not be experts in using the software; specific directions are given for each task, and different activities assume different levels of comfort with the software. Side notes are given throughout to clarify tasks that may have been used in previous activities, yet some may need more clarification. The activity notes often include cautions about tools or may include guiding questions to begin the activity.

These two resources—beneficial for introducing concepts, strengthening understanding, and reviewing—will make a nice addition to any GSP library.

—Lori Rapp
North Carolina Virtual Public Schools
Smethport, PA

A Guide to Groups, Rings, and Fields, Fernando Q. Gouvea, 2012. 327 pp., $49.95 cloth. ISBN 978-0-88385-355-9. Mathematical Association of America; www.maa.org.



This collection of definitions and results relating to groups, rings, and fields is a complete compendium. It discusses in great detail many of the various results about these marvelous objects.

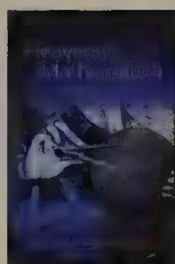
The range of material covered is extensive. The book does not include proofs of the various theorems, but proof is not its purpose. Its purpose is to cover and review material from a standard modern algebra graduate course sequence.

This is not a book for use in a standard K–12 setting or, in fact, in most undergraduate settings This book is for graduate students preparing to take qualifying exams in algebra as they work toward their doctorate in mathematics or a mathematics-related field.

I heartily recommend this book to any graduate student preparing for exams. It may not be as beneficial for other readers.

—David Royster
University of Kentucky
Lexington, KY

Heavenly Mathematics: The Forgotten Art of Spherical Trigonometry, Glen Van Brummelen, 2012. 216 pp., $35.00 cloth. ISBN 978-0-691-14892-2. Princeton University Press; www.press.princeton.edu.



This interesting historical development of spherical geometry and spherical trigonometry is designed for anyone with knowledge of plane geometry and plane trigonometry. The author's use of chronological historic events makes the need for this area of mathematics crystal clear. His arguments and proofs connect planar concepts to spherical concepts with a skill and clarity that makes the relationships almost obvious.

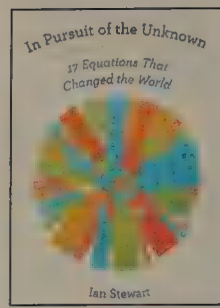
Van Brummelen expertly uses historical knowledge of the contributions of people and cultures to craft a coherent, smoothly flowing story. The detailed explanations of how Ptolemy created his table of chords and of why Napier's logarithms made his work on spherical trigonometry manageable, for example, seem reasonable. Each chapter ends with exercises.

Applications for spherical trigonometry are made as easy for readers to understand as the proofs. Skillful sketches, copies of diagrams, and excellent photographs serve to enhance the explanations presented. Stereographic projections and navigation are two developed applications.

This book could serve as an excellent textbook for any secondary school mathematics classroom at or above the level of geometry and certainly trigonometry; as the basis for a high school honors class; or as a textbook and seminar topic for college students.

—Teresa Floyd
Mississippi College
Clinton, MS

In Pursuit of the Unknown: 17 Equations That Changed the World, Ian Stewart, 2012. 352 pp., $26.99 cloth. ISBN 978-0-465-02973-0. Basic Books; www.basicbooks.com.



At first glance, *In Pursuit of the Unknown* seems to be an overachieving answer to a mathematical party game: "Name the three most important equations in history." The book does provide Stewart's answer to that question, but it also gives readers a compact history of some pivotal mathematical results and their implications.

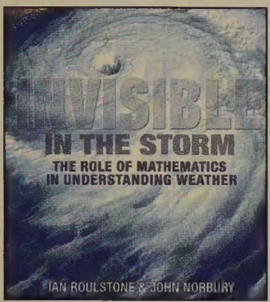
Each chapter begins with a statement and explanation of an equation and then traces some historical developments that Stewart sees as consequences of the equation. The equations range from the so-called Pythagorean theorem to the financial Black-Scholes equation; eight of the equations come from physics. Strictly speaking, not all of Stewart's seventeen "equations" are equations. One (the second law of thermodynamics) is an inequality; another (the derivative) is a definition; and a third (Maxwell's equations) is a quartet of equations.

Quibbles about what constitutes a single equation notwithstanding, *In Pursuit of the Unknown* is worthwhile reading for mathematics teachers and interested students. I found myself wondering why some equations such as Zipf's law, $P = NP$, the general quintic equation, and Bayes's formula were omitted. Readers may wonder what other equations could have been included.

Stewart's chains of reasoning from an equation to its ramifications are marvelous excursions into mathematical history and commentary on contemporary mathematics. He is an accomplished writer who is able to weave verbal tapestries from his chosen elements. I recommend *In Pursuit of the Unknown* to mathematics teachers and ask, "Which equations would you choose?"

—James V. Rauff
Millikin University
Decatur, IL

Invisible in the Storm: The Role of Mathematics in Understanding Weather, Ian Roulstone and John Norbury, 2012. 346 pp., $35.00 cloth. ISBN 978-0-691-15272-1. Princeton University Press; www.press.princeton.edu.



As a mathematics educator, I have often wondered how people think about the probabilities that accompany local weather forecasts. What might it mean when a meteorologist says that there is a 70 percent chance of showers? Although *Invisible in the Storm* did not answer my question about how forecasts are understood, it provided a coherent and entertaining narrative for understanding how and why modern forecasts are constructed.

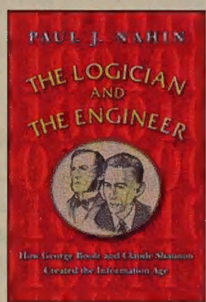
Roulstone and Norbury detail how analyses of models that were first developed from scientific observations accompanied mathematical breakthroughs that were subsequently used to construct, test, and refine new forecasting models. By focusing on the chronological development, the authors exemplify the general cycle between empirical and theoretical results that leads to progress in applied mathematics. While emphasizing the interplay among mathematics, physics, philosophy, sociology, and technology, they also provide important insights about the personalities of seminal contributors such as Bjerknes, Richardson, and Charney.

The majority of the text would be accessible to secondary school teachers. The more advanced mathematics is set off in technical sidebars and is not necessary for grasping the central ideas; explanatory figures and text accompany basic physics concepts; meteorological jargon is carefully explained and illustrated. I recommend *Invisible in the Storm* both to mathematics undergraduates and educators who are interested in applied mathematics, weather forecasting, or both.

—Steven Boyce
Virginia Tech
Blacksburg, VA

The Logician and the Engineer: How George Boole and Claude Shannon Created the Information Age, Paul J. Nahin, 2012. 244 pp., $24.95 cloth. ISBN 978-0-691-15100-7. Princeton University Press; www.press.princeton.edu.

From Aristotelian logic to quantum computing, this ambitious book shows how the pioneering genius of both George Boole and Claude Shannon laid the foundations for the computer age. Boole, a nineteenth-century mathematician, applied symbolic algebra to logical reasoning. Shannon, an electrical engineer born more than fifty years after Boole's death, created electrical circuitry using Boole's algebra of logic. Today we take information processing for granted, with little appreciation for the profound insight required to conceive of and build machines that reason. This book shows us the people and the processes that made these advancements possible.



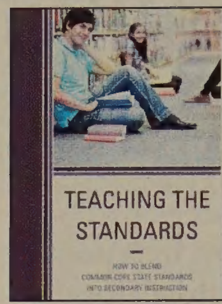
Biographical information about both Boole and Shannon is provided to humanize their accomplishments, but the heart of the book is the explanation of the logic of digital circuitry. AND, OR, NOT, NAND, NOR, and XOR gates are described in detail. Using these basic building blocks, the author shows how to create information-processing circuitry. Turing machines and computational complexity are explored, and the book concludes with a consideration of future developments in computing.

Knowledge of electronics is not a prerequisite for understanding the circuitry, but an appreciation for analytical reasoning is essential; knowledge of algebra, combinatorics and probability, and matrix multiplication are also needed. Even with some background in electronics and extensive mathematical knowledge, I found some of the circuit development confusing.

This book is not light reading. It would be excellent for advanced high school juniors or seniors with a strong interest in computer science as well as mathematics.

—Tom Ottinger
Reinhardt University (retired)
Waleska, GA

Teaching the Standards: How to Blend Common Core State Standards into Secondary Instruction, Harriet D. Porton, 2013. 192 pp., $27.95 paper. ISBN 978-1-4758-0333-4. Rowman and Littlefield Publishing Group; www.rowman.com.



The author approaches teaching and learning of the Common Core State Standards through the lens of literacy for the four main subject areas of mathematics, science, English, and social studies. Key ideas include using cooperative learning, the relationships and applications of neuroscience and learning theories, and the habits of mind (for mathematics, the Standards of Mathematical Practice).

The book has three sections, with the second focusing on content-specific issues regarding literacy. Overall, the stated purpose is for teachers, administrators, preservice and in-service instructors, or professional development leaders to view the Common Core State Standards in action.

The text emphasizes the importance of having challenging, relevant lessons for all students while including the emotional side of learning mathematics. Strategies are given for teaching reading and writing along with vocabulary. The given lesson-plan template suggests a direct-instruction approach rather than a plan structured to encourage problem solving, discovering mathematics, and thinking creatively. What is lacking is an example and explanation of a lesson plan that incorporates the Standards of Mathematical Practice and challenging content, reading, and writing. How does a teacher put it all together?

Teaching the Standards is a good resource for understanding how the Common Core State Standards evolved and a good overview of reading and writing strategies for mathematics learning and teaching.

—Pamela Bailey
George Mason University
Fairfax, VA

MY FAVORITE lesson

Arsalan Wares

Paper Folding Promotes Mathematical Thinking

A great problem allows us to discover and apply its underlying structure to go beyond the specific cases and scenarios in the original problem. When solved, a great problem provides us intellectual gratification as well as a sense of learning and, perhaps, bewilderment. I use the problem presented here as a part of one of my favorite lessons, a paper-folding activity that focuses on perimeter.

Consider a strip of paper with a perimeter of 22 inches (see **fig. 1a**). After we fold the paper once (see **fig. 1b**), the perimeter of the shape is $20 + \sqrt{2}$ inches. After a second fold (see **fig. 1c**), the perimeter of the new shape is now x inches. After a third fold (see **fig. 1d**), the perimeter of the resulting shape is y inches. Each fold forms a right angle between two parts of the strip.

Task 1: Determine lengths x and y.

Task 2: The original strip has 4 sides (see **fig. 1a**), the first folded shape has 7 sides (see **fig. 1b**), and the second folded shape has 10 sides (see **fig. 1c**). Explain why each fold adds 3 additional sides. How many sides would the shape have after n folds?

Let's discuss task 1. Suppose that the width and the length of the original rectangle were a and b, respectively. Because the perimeter of the original rectangle is 22 inches, we know that $2a + 2b = 22$ (equation 1).

Although we lose $2a$ from the perimeter of the previous shape (the part of the original perimeter that has been "covered" by the fold), we also add $\sqrt{2}a$ to the perimeter (the length of the crease we create by the first fold). To see this, focus on the right triangle where the paper overlaps. The two legs of this right triangle are eliminated from the perimeter of the previous shape. However, the hypotenuse of this right triangle becomes a part of the perimeter of the new shape. So for the first folded shape, the perimeter is $2a + 2b - 2a + \sqrt{2}a = 20 + \sqrt{2}$, or $2b + \sqrt{2}a = 20 + \sqrt{2}$ (equation 2).

Solving equations (1) and (2) for a and b, we determine that $a = 1$ and $b = 10$. We know a and b, and we recall that x represents the perimeter after two folds and that y represents the perimeter after three folds. Thus, we can reason as we just did for the first fold to calculate lengths x and y:

$$\begin{aligned} x &= \left(20 + \sqrt{2}\right) - 2a + \sqrt{2}a \\ &= \left(20 + \sqrt{2}\right) - 2(1) + \sqrt{2}(1) \\ &= 18 + 2\sqrt{2} \end{aligned}$$

and

$$\begin{aligned} y &= \left(18 + 2\sqrt{2}\right) - 2a + \sqrt{2}a \\ &= \left(18 + 2\sqrt{2}\right) - 2(1) + \sqrt{2}(1) \\ &= 16 + 3\sqrt{2}. \end{aligned}$$

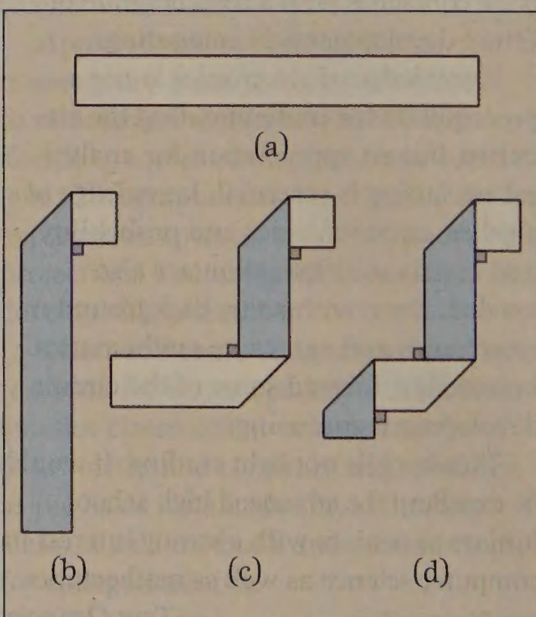


Fig. 1 A long strip of paper folded several times prompts questions about perimeter.

Now we address task 2. Each fold causes two of the sides to split, and the hypotenuse of the overlapping right triangle becomes an additional new side. Thus, each new fold causes the number of sides to increase by 3. After n folds, the shape will have $4 + 3n$ sides.

One strength of this problem is the relative ease with which secondary school students understand and model the problem, yet the solution poses many challenges. Task 2 provokes students to explore an arithmetic sequence and generates a powerful connection among geometry topics—perimeter and paper folding—and topics in algebra—arithmetic sequences and systems of equations. Because of the hands-on nature of the problem, students are better able to communicate their mathematical thinking to their peers.

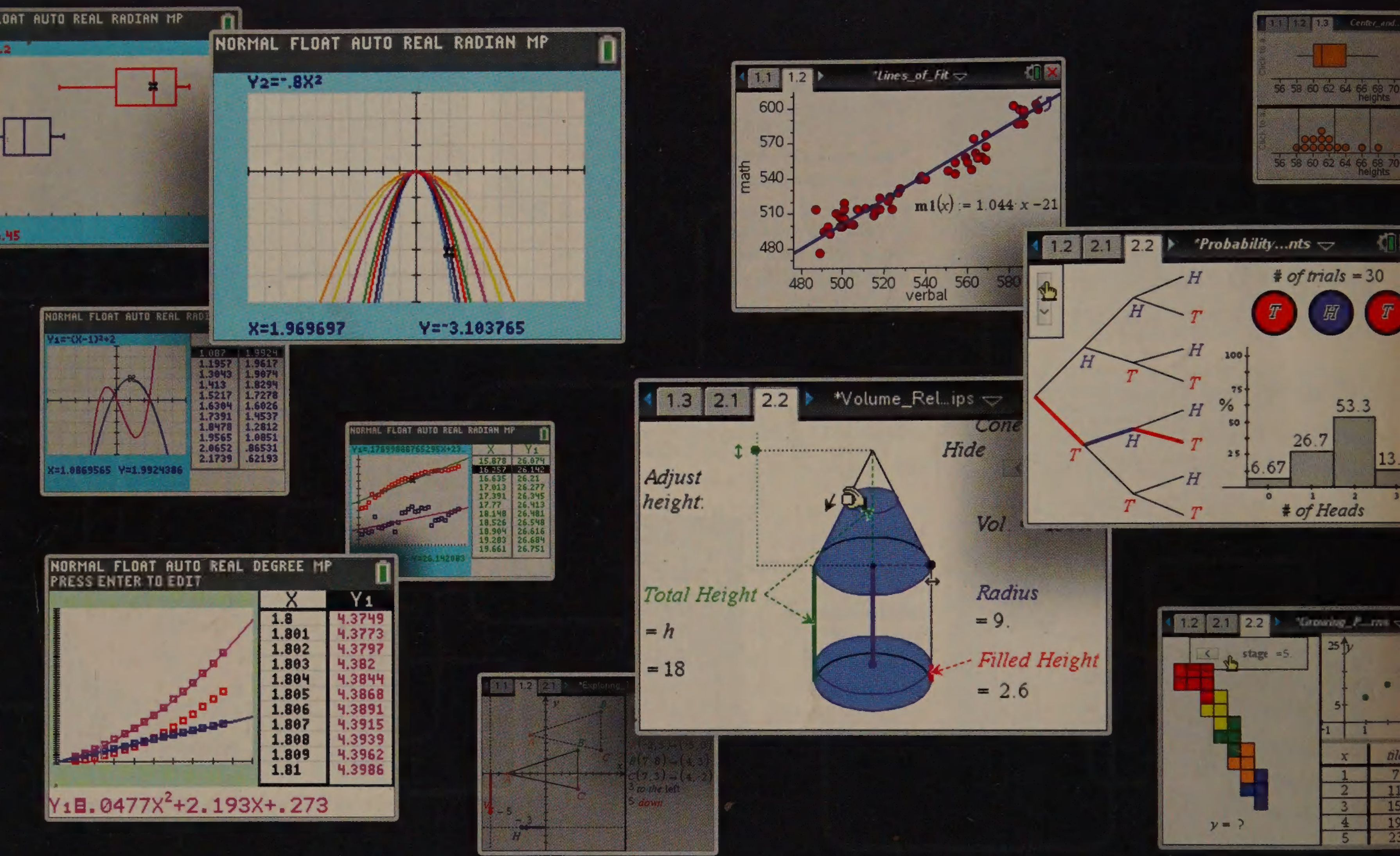
This problem and the discussion of students' solutions can easily be transformed into an invigorating lesson on perimeter. It may even become *your* favorite lesson!

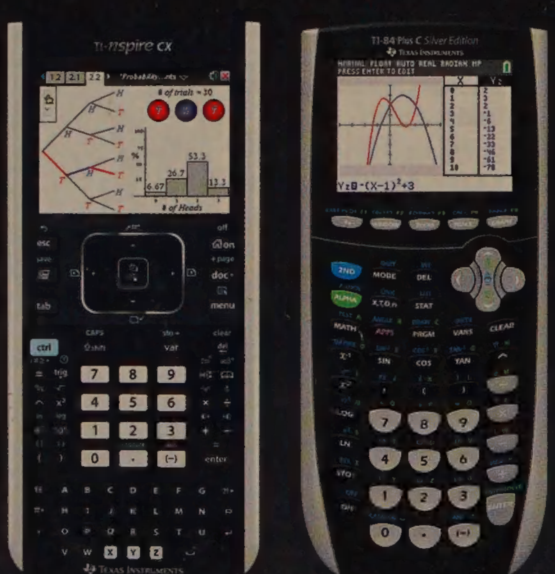
ARSALAN WARES, awares@valdosta.edu, teaches mathematics education courses at Valdosta State University in Georgia. He enjoys giving professional development workshops.

The Back Page provides a forum for readers to share a favorite lesson. Lessons to be considered for publication should be submitted to mt.msubmit.net. Lessons should not exceed 600 words and are subject to abridgment.

Department editor
Roger Day, *day@ilstu.edu, Illinois State University, Normal, IL*

THOUSANDS OF STANDARDS-ALIGNED MATH ACTIVITIES





TI-Nspire™ CX

TI-84 Plus C Silver Edition

Enhance your math curriculum with our free, ready-to-use, standards-aligned classroom activities for middle grades through high school. Activities include student worksheets and teacher notes to support standards.

Developed by educators for educators, these activities allow students to visualize concepts clearly and make faster, stronger connections. Use "as is" or customize to meet your curriculum needs, it's completely up to you.

Download activities now at **education.ti.com/activities**.

Texas Instruments